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**The Work of Ambjörn Naeve
within the Field of Mathematics Educational Reform**

Ambjörn Naeve



CID
Centre for
User Oriented IT Design

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1. Problem Statement

During the last two decades, the spectacular advancements within the field of information technology have created powerful graphical workstations with possibilities to study mathematics in new and exiting ways. Today it is evident how computer based animations and simulations have affected most fields that involve mathematical applications in some way. Visualization of structural relationships and dynamic processes has emerged as a field of its own, with applications within science and technology as well as within economics and social sciences.

It is therefore something of a paradox that one of the fields that seems to have been least affected by this development is the field of mathematics itself. This is especially true of mathematics education, i.e. mathematical didactics.

At our universities we are still carrying on the traditional ways of mathematics teaching – executing our courses in the overall spirit of “the same procedure as last year”! Among mathematics teachers, computers are often considered a threatening element, that focuses the students’ attention in the wrong direction. This attitude is easily reinforced by the multitude of low-quality educational software that supports mathematically trivial pursuits in one way or another.

In fact, this deplorable state of affairs has caused the issue of computer-supported or computer-disturbed mathematics to surface as a major discussion theme among mathematical educationalists.¹

Of course, there are many exceptions from this basic pattern in the form of individual teachers that involve themselves in trying to make intelligent use of the possibilities of pedagogical renewal that are offered by the emerging ICT technology. But as long as they act alone, as isolated enthusiasts emerged in an ocean of skeptics, their efforts and experiences will remain hidden and hence be difficult to harness and reuse in a systematic way.

¹ See the report titled *Datorstödd eller datorstörd matematikundervisning?*, Högskoleverkets skriftserie 1999:4S, ISSN 1400-9498.

2. Background

During more than 3 decades Ambjörn Naeve has been involved in mathematics education at KTH – both as a teacher in the Mathematics department, and as an educational reformist. Over the past 15 years, Naeve has initiated and coordinated a number of projects aiming to make use of computers in order to increase the comprehensibility and accessibility of mathematical concepts and structures at all levels of the educational system. These projects have all been based on his firm conviction that increasing the students' intuitive understanding of mathematical structures – both at the university level and at the more elementary school level – is a key element in motivating them to pursue mathematical studies in general.

This work originated as a part of Naeve's research in the field of geometric modeling within the Computer Vision and Active Perception (CVAP) research group at NADA. During the last 4 years it has been a part of Naeve's research work in the field of interactive learning environments at the Centre for user-oriented IT Design (CID) at NADA.

The work has resulted in a number of software tools for the interactive exploration of mathematics [12], several of which have attracted both national and international attention.

Some of the more recently developed ones are PDB (Projective Drawing Board), Conzilla (<http://cid.nada.kth.se/il/conzilla/default.html>) and CyberMath (<http://www.nada.kth.se/~gustavt/cybermath/>). They were presented last year at the Siggraph conference in New Orleans – probably the most prestigious computer conference in the world. CyberMath has also been accepted for presentation at ICDE-2001 (the 20:th World Conference on Open Learning and Distance Education) in Düsseldorf in April (<http://www.icde.org>). Conzilla and CyberMath have been partly developed within the APE (Archives, Portfolios, Environments) project in cooperation between CID and the Swedish Learning Lab (<http://www.learninglab.kth.se/library/presentations>).

The projects have involved systems design and programming by Naeve himself, but he has also headed a number of development teams involving 3rd and 4th year students at the Computer Science department (NADA) as well as projects for the masters thesis [19], [21], [23] and for the doctoral thesis [20]. The common theme of all these projects has been the presentation of mathematical ideas and structures in a way that facilitates an increased understanding of them by making it possible to explore them interactively through various forms of experimentation and visualization.

Initially, the projects were focused on geometry – which resulted in software programs like MapCon (1986), MacWallpaper (1987) and MacDrawboard (1988), but programs like HyperFlow (1987) and MapAnalyze (1989) which dealt with the general concept of function and PrimeTime (1992), which dealt with elementary arithmetic were also created. See [12] for further details on these software tools.

During recent years the work has become directed towards the creation of computer-supported mathematics tools that can function together in a modularized and distributed interactive learning environment.

In this respect, the following projects deserve to be mentioned:

- Geometric Toolbox (research work at CVAP, 1986-1994).
- The Garden of Knowledge (interdisciplinary project at CID since 1996).
- Projective Drawing Board (doctoral work at CVAP, 1995-1999).
- Conceptual browsing (started as a masters thesis work at CID 1998-1999, presently a doctoral thesis project at CID).

Below, some of these projects will be described briefly. For a more detailed description the reader is referred to [12].

2.1. The CVAP research group

The Computer Vision and Active Perception (CVAP) group is an internationally renowned research group at NADA within the field of Computer Vision and Robotics, which has been developed during the past 15 years under the leadership of Prof. Jan-Olof Eklundh. The CVAP group has a strong foundation in mathematics and computer science. It has acquired an international reputation for its ability to transform mathematical ideas into technically viable applications, which has made the group able to attract a large number of gifted students. Over the years, several of the so called "excellence positions" for doctoral students at KTH have been held by research students at CVAP. In combination with the inspiring leadership of Prof. Eklundh, this has made possible some mathematical software design of the highest international quality like e.g. Geometric Toolbox (Naeve & Appelgren 1986-94) and Projective Drawing Board (Naeve & Winroth 1995-1999).

2.2. The Geometric Toolbox project

The typical geometric modeling situation of today is characterized by - and quite frequently plagued by - a number of tools with a high degree of special "stream-lined" performance. This has almost invariably led to "ad hoc" choices and simplifications that have created mathematical inconsistencies and thereby rendered almost all of the tools incompatible with the others -- preventing them to work together in a coherent fashion against the same "all inclusive" universal geometric background.

.....

At CVAP , Ambjörn Naeve and Johan Appelgren have developed a software package called Reflections [19], which is a system for the interactive study of surface shape. This system was used as an experimental platform for the theory developed by Naeve in his dissertation [7]. Reflections is part of a

software system called Surface-Geometry, which is a mathematically based, computationally efficient geometric representation scheme for 3D surface [12].

The Surface-Geometry system is itself part of a larger geometric modeling project within CVAP, called Geometric Toolbox, which is aimed at producing an interactive "mathematics-friendly" geometric experimentation environment - a kind of geometric "object library" - consisting of a collection of compatible and reusable geometric structures and algorithmic components. Using this kind of geometric toolbox, different kinds of geometric experiments - of relevance in such fields as e.g. computer graphics, computational geometry and computer vision - can be easily "wired together" and all the relevant parameters can be manipulated in a mathematically controlled and interactively observable way.

The desire to perform such experiments - where one is combining "heavy computing" with "immediate viewing" of the result - is growing rapidly within the community of computational geometry - as the power of such techniques in developing and testing different algorithms is becoming more and more apparent. This is due to a combination of the enormous increase of computational power that has manifested itself in hardware components over the last few years and the advanced graphics workstation capabilities that are on the verge of settling down on everybody's desktop. It has finally become feasible to simulate a large class of complicated geometrical situations and obtain information online with direct relevance to the understanding of the underlying problem. The possibility to interactively expand ones intuition about a problem - by performing mathematically controlled experiments in this way - is a very powerful technique that is bound to have a profound effect on the entire research methodology of the future.

For a detailed description of the results of the Geometric Toolbox project, the reader is referred to [6], [7], [12] and [19].

2.3. Projective Drawing Board: dynamic geometric explorations

PDB (Projective Drawing Board) is a program that supports interactive exploration of geometric constructions in the projective plane². The projective plane is an enlargement of the ordinary (Euclidean) plane which is constructed by introducing new elements (a set of ideal points and one ideal line) in such a way that two parallel lines intersect in an ideal point and all the ideal points lie on the ideal line.

In the projective plane, two lines always intersect in (= lie on) one point, and of course two points still lie on (= intersect in) one line. Hence the structural relationships between points and lines become much simpler, since they are

² PDB has been created by Harald Winroth as a part of his doctoral project [20] at CVAP under the supervision of Ambjörn Naeve. The program is based on an earlier prototype called MacDrawboard, which was developed by Ambjörn Naeve and a group of Computer Science students in 1998 [12].

now devoid of the classical euclidean exceptions caused by parallel lines, that can lead to complicated combinatorial problems.

In the projective plane, points and lines are in fact represented by the same algebra, and it is only the interpretation of the algebraic formulas that determine their graphical appearance (as a point or as a line).

Every geometric construction has a history, which reflects the order in which the construction has been built up. A construction process can be regarded as an interplay between random choice (e.g. choose two points P and Q) and canonical necessity (e.g. draw the line PQ). A geometric object can partake of both these elements (e.g. choose a line on P). To any geometric object we can therefore associate a set of children and a set of parents in a natural way. In the example above, the line PQ is a child of both the point P and the point Q , and both of these points are parents of the line PQ .

One of the basic ideas in PDB is to keep track of the history of a geometric construction and make it possible to change it in a consistent way. This means that a change that affects an object a certain stage in the construction should propagate forward so that it affects all the children of this object. In order to update the construction in this way, PDB has access to an entire hierarchy of coordinate systems that keep track of the position of each object relative to its parents.

Presently PDB works only with the elements of classical projective geometry, i.e. points, lines and conics. However, the system is designed in accordance with the object-oriented paradigm and it is well modularized to make it easy to enlarge and expand in various ways.

PDB presents both a graphic and a logic view of a geometric construction. Moreover, the program allows you not only to change the position of an object, with coherent updates of the effects on its children, but to actually change the logic of the construction by dynamically changing the constraints of an object. For example, a point which is a child of a conic can be torn off the conic and either be turned into a free point, or be subjected to some other constraint and e.g. become a point on a some line. This allows you to play with a construction and experience precisely under what conditions certain things happen, i.e. you can interactively explore the if-and-only-if conditions of a geometric theorem.

Graphic view

Logic view

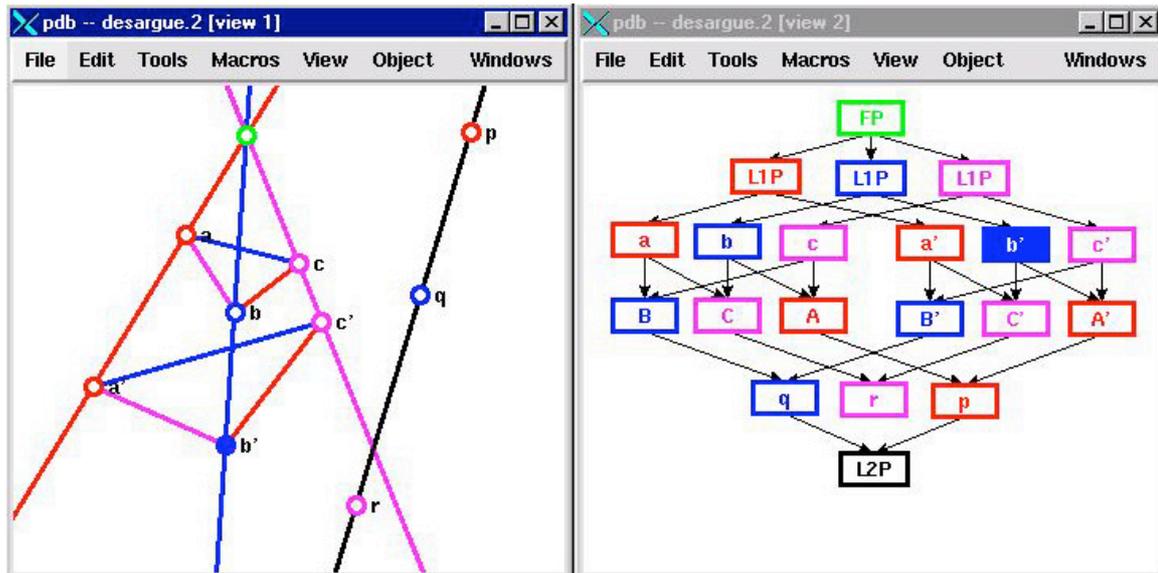


Fig. 1: Desargue's theorem. The corners of two triangles (abc and $a'b'c'$) are perspective from a point (green) if and only if the corresponding lines (ab & $a'b'$, bc & $b'c'$ and ca & $c'a'$) of the triangles are perspective from a line (black), i.e. the points p , q , r are collinear.

To convey an idea of the dynamic possibilities of PDB, a QuickTime movie that illustrates the dynamic exploration of Desargue's theorem is available at <http://www.nada.kth.se/~amb/SnapzPro/Desargues.mov>.

2.4. The Centre for user-oriented IT Design (CID)

CID, which was started in 1995, is a competence center at the department of Numerical Analysis and Computing Science (NADA) at KTH. Under the leadership of Prof. Yngve Sundblad, it has developed into a fertile meeting ground for a unique mix of competencies that includes researchers from computer science, mathematics, aesthetics, cognitive science and pedagogy, who have teamed up in order to perform research on a variety of issues related to the field of Human Computer Interaction. See <http://cid.nada.kth.se> for a detailed presentation of the CID research environment.

Today, CID is engaged in 4 different areas of research:

- Connected Communities (= Digital Worlds).
- Interactive Learning Environments.
- New forms of Interaction.
- User Orientation.

The goals and characteristics of CID can be summarized in the following way:

- to integrate usability with technical and aesthetic aspects
- to create an attractive environment at KTH for strong cooperation between

- academy, industry and users.
- to produce “pre-competitive” results in the form of prototypes, demonstrators and user studies.
 - to engage in strong international collaboration.
 - to make strong contributions to education.

2.5. The Garden of Knowledge project

The Garden of Knowledge project was started at CID in 1996 by Ambjörn Naeve as an effort to support a cross-disciplinary understanding of different concepts and phenomena by describing their mutual relationships as well as their evolutionary development processes through different time periods and cultures. The prototype that has been developed explored the concept of symmetry as a way to describe some of the fundamental connections between mathematics and music – ranging from the ideas of Pythagoras to the electronic synthesizer.

(<http://cid.nada.kth.se/il/gok/default.html>)

The experiences gained from the Garden of Knowledge project [11] have contributed to the decision at CID to widen this field of study to encompass research on Interactive Learning Environments (ILEs) in general. An important goal in this work is to contribute to the development and dissemination of design principles for ILEs on the Internet that are distributed and modularized in compliance with emerging international standards.

CID is therefore continuously monitoring such standardization projects (IMS, LOM, RDF, ...). Present work is focusing on mathematics (geometry) and literature (Strindberg) as two different test beds for content, with the aim to come up with design principles for technically, aesthetically and pedagogically well-structured resource components that can be archived, retrieved and reused within such a standardized framework.

In a longer perspective, the goal is to produce an open-source prototyping tool for a modularized and distributed ILE, where it is possible to test and compare different principles for describing, searching, and interfacing a variety of different resource components in order to create various forms of flexible and customized ILEs that support individualized forms of inquiry-based learning.

In order to be successful, such an ILE prototyping tool must make use of a number of synergies between different kinds of competencies. At CID we have access to a large number of different competencies that are of strategic importance in this development project:

- Aesthetic competence, which has earned the interface of the Garden of Knowledge prototype international recognition (ref: Siggraph 2000) as a pleasing and user-friendly way to interact with an inter-linked and modularized learning environment with both theoretical and experimental modules.

- Pedagogical competence, which has made the content of the first GoK prototype comprehensible, accessible and stimulating for a much wider group of users than that which was originally targeted: University math students with an interest in music and vice versa.
- Computer science competence, which has made possible the use of many exiting features of the emerging web programming techniques both with regard to modularization and platform independence (Java, Servlets) as well as information markup (XML, IMS).
- Cognitive science competence, that guarantees that the human aspects are not left out in the midst of all the necessary machine handling. In this respect, the user-oriented profile of CID constitutes a strong integrative force.

2.6. Some ILE projects at CID

Within the field of interactive learning environments, CID is presently engaged in a number of projects that include the following general activities:

- **Contextual navigation and content presentation:**
Conceptual browsing with Conzilla.
<http://cid.nada.kth.se/il/conzilla/default.html>.
- **Conceptual Modeling of the Structure and Activities of Organizations:**
Accessibility issues and international standardization processes.
http://cid.nada.kth.se/il/cm/conceptual_modeling.html.
- **The Mathemagical Garden of Knowledge as a Knowledge Manifold:**
Constructing a Virtual Mathematics Exploratorium.
<http://cid.nada.kth.se/il/gok/default.html>.
- **The Swedish Learning Lab:**
Component archives, Student Portfolios, 3D Learning Environments.
<http://cid.nada.kth.se/il/swell/default.html>.

3. Concept Browsing

What follows here is a brief description of the main ideas behind concept browsing, which is a new way to structure and present information that has been invented by Ambjörn Naeve. For a detailed discussion of these ideas, see [13].

3.1. Conceptual topologies

Let S be a set of concepts, and let C be a concept in S . A *conceptual neighbourhood* of C in S is a diagram that expresses conceptual relationships between C and some other concepts from S . A conceptual neighbourhood will also be referred to as a *local context*. The conceptual topology on a concept set S is the set of all conceptual neighbourhoods (in S) of concepts of S .

3.1.1. Traditional conceptual topologies

Presenting informational content requires some form of containing structure - or context - for the information that is to be presented. A traditional dictionary, for example, uses lexicographic ordering of the labels representing the content to create the structure of the presentational context. This lexicographic context has the advantage of making the content easily accessible through the corresponding label, but at the same time it has the drawback of not showing any conceptual relationships between the different pieces of content.

Therefore, a dictionary creates a totally disconnected (= discrete) conceptual topology on the set of the corresponding components - with each separate component corresponding to an isolated concept.

A textbook, on the other hand, normally makes use of some form of taxonomy (= classification scheme) in order to create a suitable context for the presented information. For example, if the textbook is about animals, they might be presented as a taxonomic type-hierarchy of insects, fish, reptiles, birds, marsupials and placentals on the first sublevel. Each of these types would then in turn be appropriately subtyped according to the level of presentation and targeted reader profiles. The chosen classification scheme creates a context that gives a relational structure to the informational content, and this context reflects the corresponding taxonomic connections between the various information components. In this way a textbook creates what could be called a taxonomically connected conceptual topology on the set of components.

3.1.2. Dynamic conceptual topologies - hyperlinked information systems

Of course, the components of a book are frozen into a single context by the order in which they are presented in relation to each other. In the case of a hyperlinked multimediated system - such as e.g. the www - the situation is very different. Here there are in general many different contexts for the components, and both their number and their form are constantly changing by the addition and removal of pages and links.

For example, a web-browser maintains a dynamic conceptual relationship between the page that is viewed now (= this page) and the page that was viewed the moment before (= the previous page). The corresponding dynamic conceptual neighbourhood is traversed by using the browser buttons 'back' respectively 'forward'. Another (larger) example of a dynamic conceptual neighbourhood is given by the browser's history list.

In fact, each web-page functions both as a container of its content and as a context for the contents that are reachable (by a mouse-click) from it. Consider a typical web-page Q. Each web-page P from which Q is reachable forms a context for Q. If Q contains a link to another web page R, then Q forms a context for R, and if R contains a link to Q, then the relationship is reversed and R forms a context for Q.

In this way the underlying link structure leads to an inextricable mixture of content and context - creating what could be termed a reachability connected conceptual topology on the set of components.

This tends to make each web-page more self-contained, and could be expected to favour a contextual design that focuses more on various forms of eye-catching techniques (animational eye-magic) than on illuminating the conceptual relationships. Of course, when designing a conceptual presentation system - as in fact when designing any kind of system - the overall aim is to use visual techniques in order to support the underlying conceptual context, and not as a substitute for this context.

The basic presentation tool for information on the web is called a web-browser. The most commonly used web is the World Wide Web, and the two dominating www-browsers are *Netscape Navigator*[®] and *Microsoft Explorer*[®]. It will be assumed that the reader has been exposed to one (or both) of these browsers, and therefore is familiar with their basic functioning.

3.1.3. Problems with the above mentioned conceptual topologies

The conceptual topologies that were discussed above are extreme in terms of their relationship between content and context. Books are totally (= linearly) ordered and do not allow reuse of components in different contexts. Hence the context of a book is fixed. The www, on the other hand, presents a totally fluid and dynamic relationship between context and content, which makes it hard to get an overview of the conceptual context within which the information is presented. This results in the all too well-known "surfing-sickness" on the web, that could be summarized as "Within what context am I viewing this, and how did I get here?"

3.2. Basic Design Principles for Concept Browsers

A conceptual organization and presentation scheme that supports the conceptual context will be referred to as a *concept browser*. The kind of

concept browser that is presented here is based on the following general design principles:

- *separate* context from content.
- *describe* each context in terms of a concept map.
- *assign* an appropriate number of components as the content of a concept and/or a conceptual relationship.
- *label* the components with a standardized data description (meta-data) scheme.
- *filter* the components through different aspects.
- *transform* a content component which is a map into a context by contextualizing it.

When designing concept maps it is important to use a conceptual modeling language that adheres to international standards. We make use of UML (<http://cgi.omg.org/news/pr97/umlprimer.html>) which has emerged during the past 5 years as “the Esperanto of conceptual modeling”. As for meta-data we make use of the IMS-IEEE proposed standard for learning objects (<http://www.imsproject.org>).

3.3. Conzilla

Conzilla is a first prototype of a concept browser, which has been developed by Mikael Nilsson and Matthias Palmér under the supervision of Ambjörn Naeve. The project started in 1998 as a masters thesis project [21], which resulted in the first version in August 1999. Since then, the development has continued, and since Sept. 2000 it has taken the form of a doctoral project where Matthias Palmér has been enrolled as a doctoral student in Ambjörn Naeve’s group at CID.

Conzilla is written in Java and uses XML as the basic format for exchanging information. Since the program is carefully designed with a clear object-oriented structure that separates the underlying logic from the presentational graphics, it can be easily adapted to different presentational styles and cognitive profiles.

Through user-feedback the interface has matured and improved, and Conzilla is now starting to attract serious attention from a number of CID’s industrial sponsors. They see Conzilla as a way to structure and present their electronically stored information in a way that improves the overview and enhances the structural connections between the different parts.

It is our intention to keep developing Conzilla as an open-source project in a Linux-like fashion. Hence it has been entered as a sourceforge project at <http://sourceforge.net/projects/conzilla>). More information about the Conzilla project can be found at <http://cid.nada.kth.se/il/conzilla/default.html> .

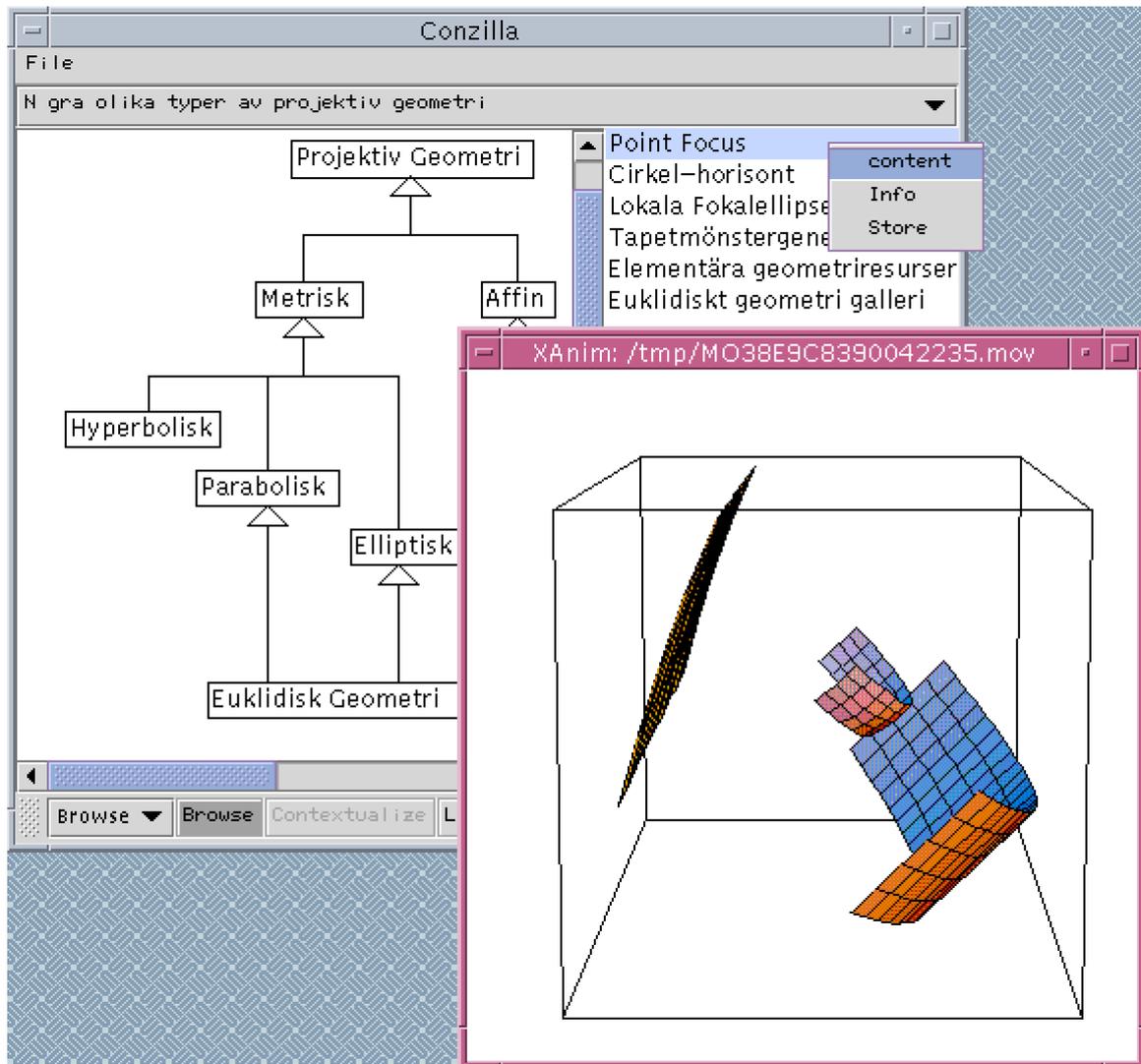


Fig. 2: An early version of Conzilla. The resource component called "PointFocus" has been located as (part of) the content of the concept "Euclidean Geometry" (Euklidisk Geometri) within the context of the concept map shown at the left of the screen-shot. By choosing "content" from the pop-up menu at the upper right, a QuickTime movie is shown in the pop-up window at the lower right. This movie is downloaded from a server and can be viewed at <http://www.nada.kth.se/~amb/SnapzPro/PointFocus.mov>.

4. Knowledge Manifolds

The conceptual framework that lies behind the Garden of Knowledge prototype is referred to as a *Knowledge Manifold* [11], [13].

A Knowledge Manifold:

- is a conceptual framework for designing interactive learning environments that support [Inquiry Based Learning](#).
- can be regarded as a [Knowledge Patchwork](#), with a number of linked [Knowledge Patches](#), each with its own [Knowledge Gardener](#).
- gives the users the opportunity to ask questions and search for [certified human Knowledge Sources](#).
- has access to distributed archives of [resource components](#).
- allows teachers to [compose components](#) and construct customized learning environments.
- makes use of [conceptual modeling](#) to support the separation between content and context.
- contains a [conceptual exploration tool \(Conzilla\)](#) that supports these principles and activities.

4.1. Creating Multiply Narrated Knowledge Components

An important part of creating a Knowledge Manifold is the design and construction of what we call *knowledge components*³ [11], [13]. They are units of multi-media based presentations of content that can be archived, referenced and used over the web. In a Knowledge Manifold, the knowledge components can be connected into learning modules (e.g. courses), by what we call *component composition*. Hence, a well designed knowledge component should present its information on several different levels of complexity, and leave it up to the user of the component to decide which level is appropriate for a specific occasion of use. This is in the spirit of e.g. a computer game, where the user is free to select the level of difficulty on which to interact with the system. The multi-level presentations contained in a knowledge component are achieved through a technique that we call *multiple narration* [13].

³ They are often referred to as *resource components*, *learning objects*, *learning offerings* or just simply *components*. We use the term “knowledge components” to emphasize that they have been created by people with knowledge. However, when they are used by a learner, they are reduced to information components. It is then up to the learner to reconstruct the knowledge that has been embedded in them, and it is a fundamental design challenge of the ILE to offer maximal support for this learning process.

A multiply narrated knowledge component could be likened to a skiing area with different slopes leading down the same mountain, each with its own level of difficulty. Moreover, the slopes are all marked in accordance with an internationally established meta-data standard. In the case of skiing, this standard is a simple colour code, and in the case of knowledge components it could be e.g. the IMS-IEEE emerging meta-data standard that is used by the ILE research group at CID.

Just as there is nothing to stop a user from skiing down a black slope even if (s)he has green prerequisite knowledge, a learner should not be prevented from interacting with a knowledge component on any level of choice. However, the system should present the learner with some form of indication of what to expect from a certain complexity level – given some information of the learner’s knowledge profile.

4.2. Composing Knowledge Components into Learning Modules

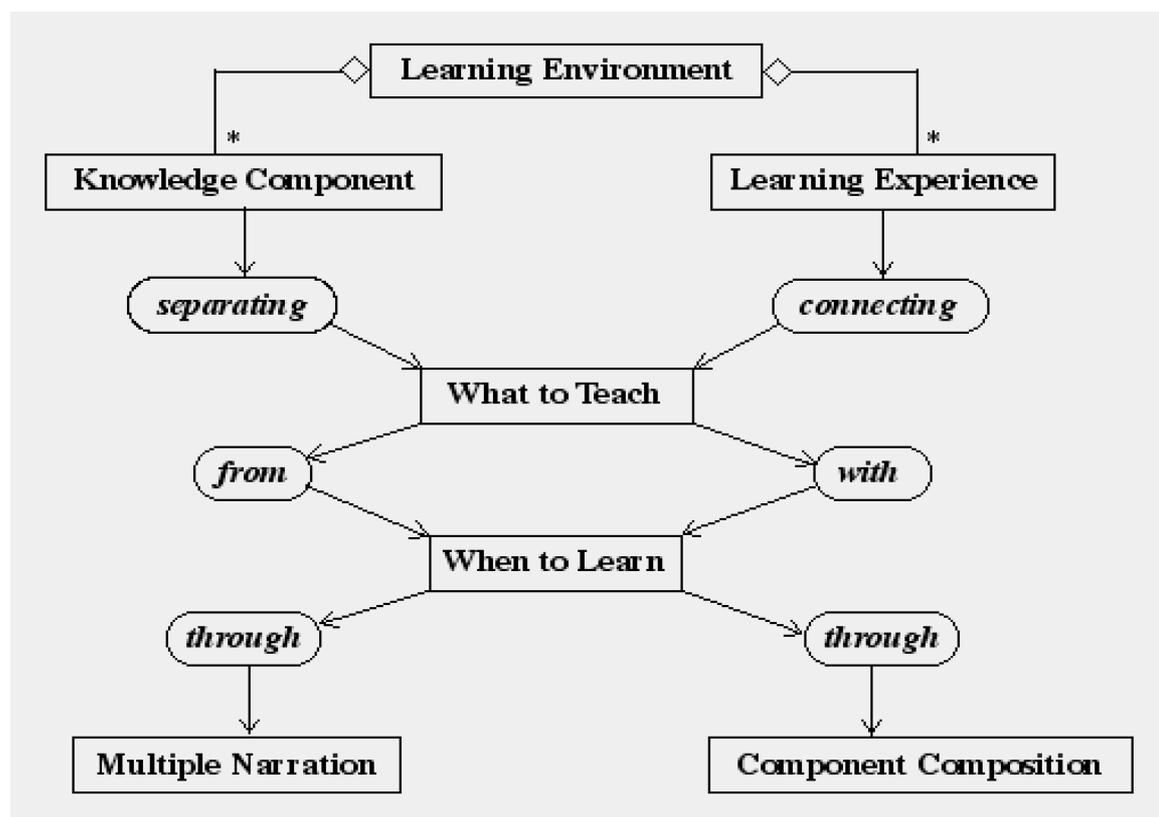


Fig. 3. Knowledge components versus Learning Experiences
 To continue the analogy with a mathematical manifold, we can say that the fibers of a Knowledge Manifold correspond to the knowledge components, and the sections in the fiber bundle of components correspond to the set of learning modules (courses) that can be constructed by connecting the components in different ways.

Hence, the overall idea is to separate the design of the components from their composition into learning modules, like e.g. courses. A knowledge component should strive to separate between "what to teach" and "what to learn", while a learning module should strive to connect these two aspects with each other in order to create customized learning experiences. The design of knowledge components is therefore fundamentally "perpendicular" to the design of learning modules, where the aim is instead to plan for a certain pre-conceived learning situation at the time when the module (or course) is composed.

It is important to emphasize that we are not interested in "freezing" live courses into computerized form (like totally videotaped courses) - except of course as an (important) means of documentation. Instead we are interested in exploring the possibilities of designing powerful and flexible knowledge components that allow living teachers to improve on the narrative of their own story, by illustrating and/or demonstrating whatever they have to say - in a more stimulating and interest-provoking way.

5. Mathematical ILE work at CID

The ongoing mathematical ILE work at CID includes the following projects:

- Using Conzilla to construct a virtual mathematics exploratorium in the form of a knowledge manifold with an atlas of mathematical contexts. The maps will be equipped with relevant meta-data, content and aspect-filters.
- Exploring new ways to study geometrical constructions by doing interactive geometry with PDB.
- Studying new ways to interact with mathematical formulas, using programs like LiveGraphics3D (Martin Krauss) and Graphing Calculator (Ron Avitzur).
- Building mathematical components archives and developing methods to interact with these components using emerging web standards (MathML).
- Exploring different ways to interact with mathematics in shared 3D interactive learning environments (CyberMath).

5.1. The Virtual Mathematical Exploratorium

5.1.1 Introduction

The *Virtual Mathematics Exploratorium* (VME) is the name of a mathematical knowledge manifold that is being constructed by Ambjörn Naeve using Conzilla. The concepts are being described with metadata and filled with content components according to the general design principles for knowledge manifolds that have been outlined above. Moreover, filters are created that allow the selective viewing of the content – based on different aspects and levels of difficulty⁴. The idea is that teachers should be able to browse through the exploratorium and find components that cover the relevant aspects of the topics they are interested in at the appropriate level of complexity. We are presently working on a component composition environment, where components will be easily integrated into customized learning modules in a work process that will support both a single “component composer” as well as a team of such composers that are involved in different forms of collaborative curriculum design.⁵

⁴ The ease with which such filters are constructed and modified is a major strength of the Conzilla tool.

⁵ This work is the major theme of a doctoral thesis project on knowledge components by Fredrik Paulsson under the supervision of Ambjörn Naeve. This project is described below.

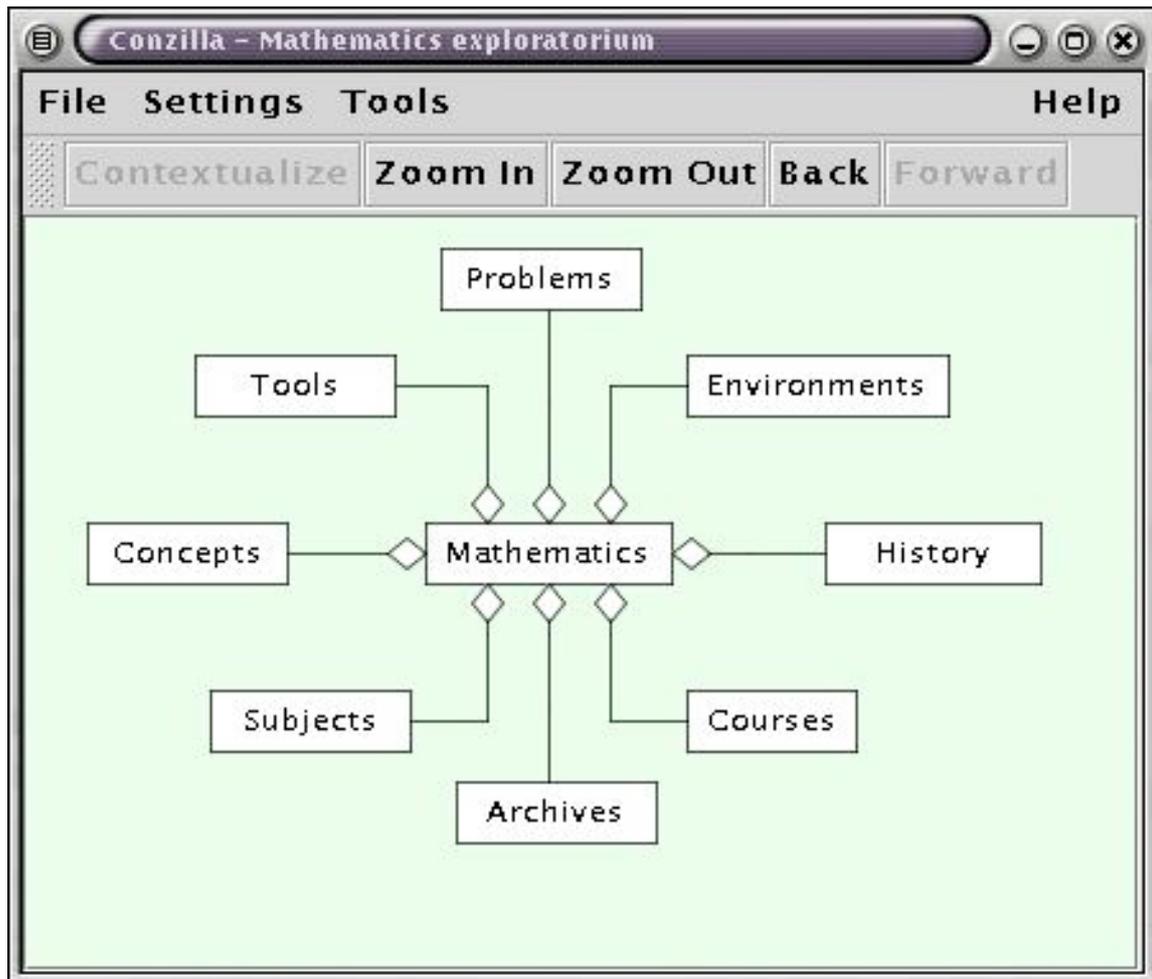


Fig. 4: The entry to the Virtual Mathematics Exploratorium.

While functioning for teachers and learning module designers as a distributed archive (open repository) of resource components, the VME will also serve as an environment where learners can navigate through the mathematical landscape and explore the topics of their own interest at the level of complexity of their own choice. This is a strategically important functionality of an interactive learning environment that aspires to support inquiry based learning. See [11] and [13] for more detailed discussions of this topic.

Figures 4 through 14 present a series of screenshots from a navigational tour of the VME. The entire VME will be available on the web within the near future.

On the top (= entrance) level of the VME, eight different parts of the concept of Mathematics are presented: Concepts, Subjects, Tools, Problems, Environments, History, Courses and Archives (Fig. 4). Of course, parts can be added (or taken out) as the design of the VME evolves.

5.1.2. Surfing the context

Double-clicking the Concepts box changes the context and takes us into the context of “Mathematics Concepts” shown in Fig. 5. In the lower right we also see a (slightly shaded) rectangle that displays metadata about the map.

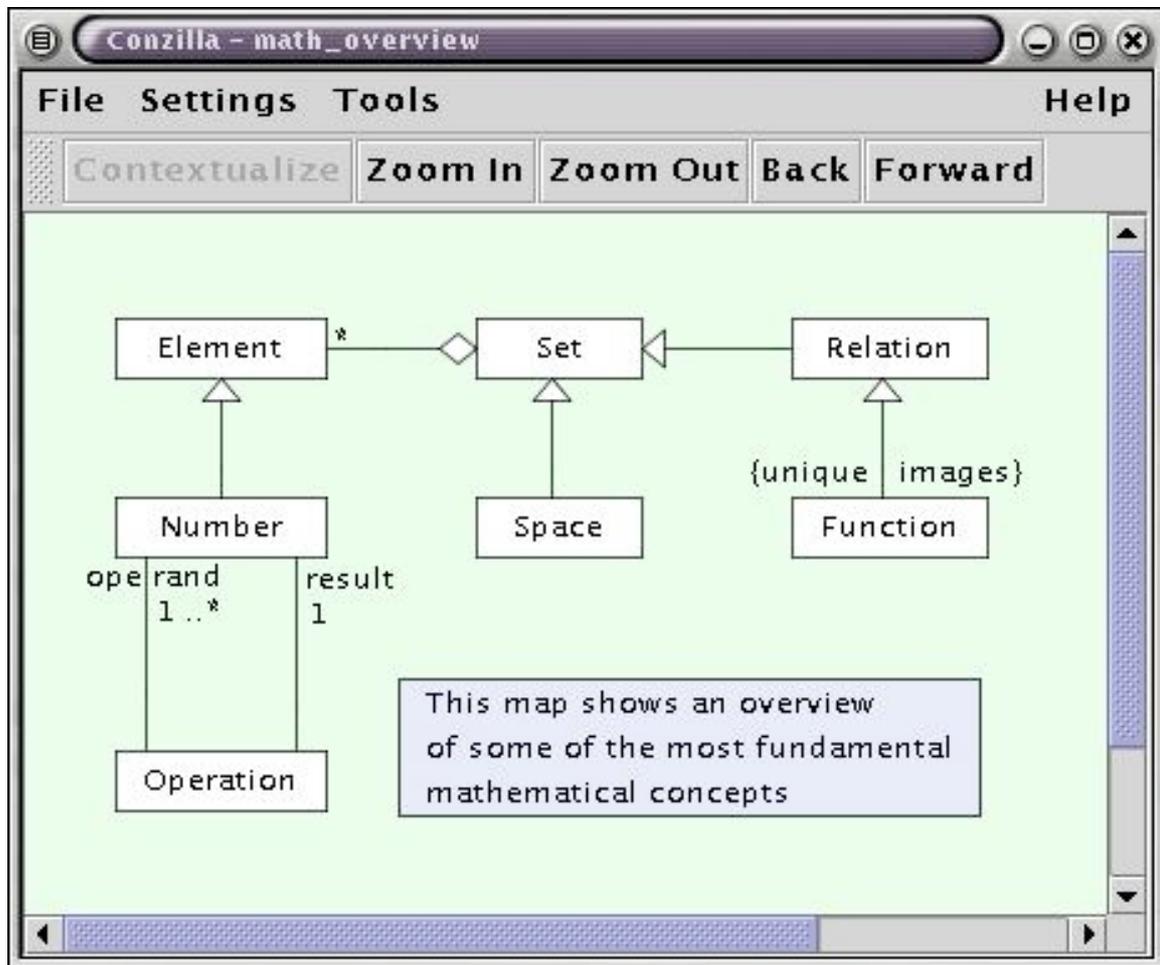


Fig. 5: An overview of mathematical concepts with a metadata description.

Double-clicking the Number-box takes us into the context of different kinds of numbers, which is shown in Fig.6. Here we see some of the different kinds (= specializations) of the concept of Number: Natural -, Integer -, Rational -, Real - and Complex number. The map also shows the relationships between these different kinds of numbers: Real numbers are displayed as specializations of (kinds of) Complex numbers, Rational Numbers as specializations of Real numbers, etc. By pointing to the specialization arrows and hitting space bar) we can bring up an explanation (= meta-data) of the structure of the corresponding specialization. As an example, pointing to the arrow between Integer and Rational would bring up the explanation: An integer is a rational number with the denominator equal to 1.

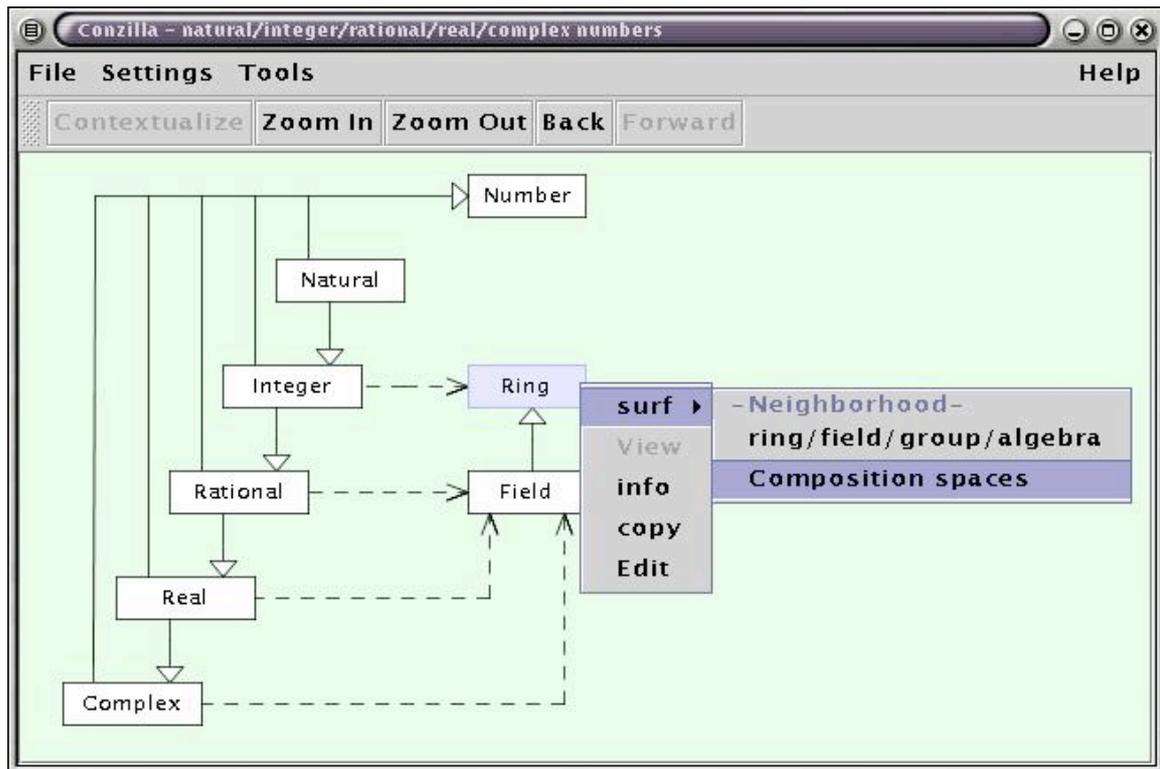


Fig. 6: Surfing the context: Switching to the map **Composition spaces**.

The dashed arrows of Fig. 6 show that the Integers are an example (= instance) of a Ring, while the Rational -, Real - and Complex numbers are examples of Fields. From the figure we also see that Field is a specialization of Ring.

Surfing by double-clicking requires a special map (called a *detailed map*) to be associated with the corresponding concept. If we want to find out more about the Ring concept, we could surf the context in a different (and more general) way. Right-clicking on “Ring” and choosing “surf “ displays a list of the *contextual neighborhood* of the Ring concept, i.e. the set of all contexts (= maps) where the concept of Ring appears. Choosing “Composition spaces” from this list brings up the map shown in Fig. 7.

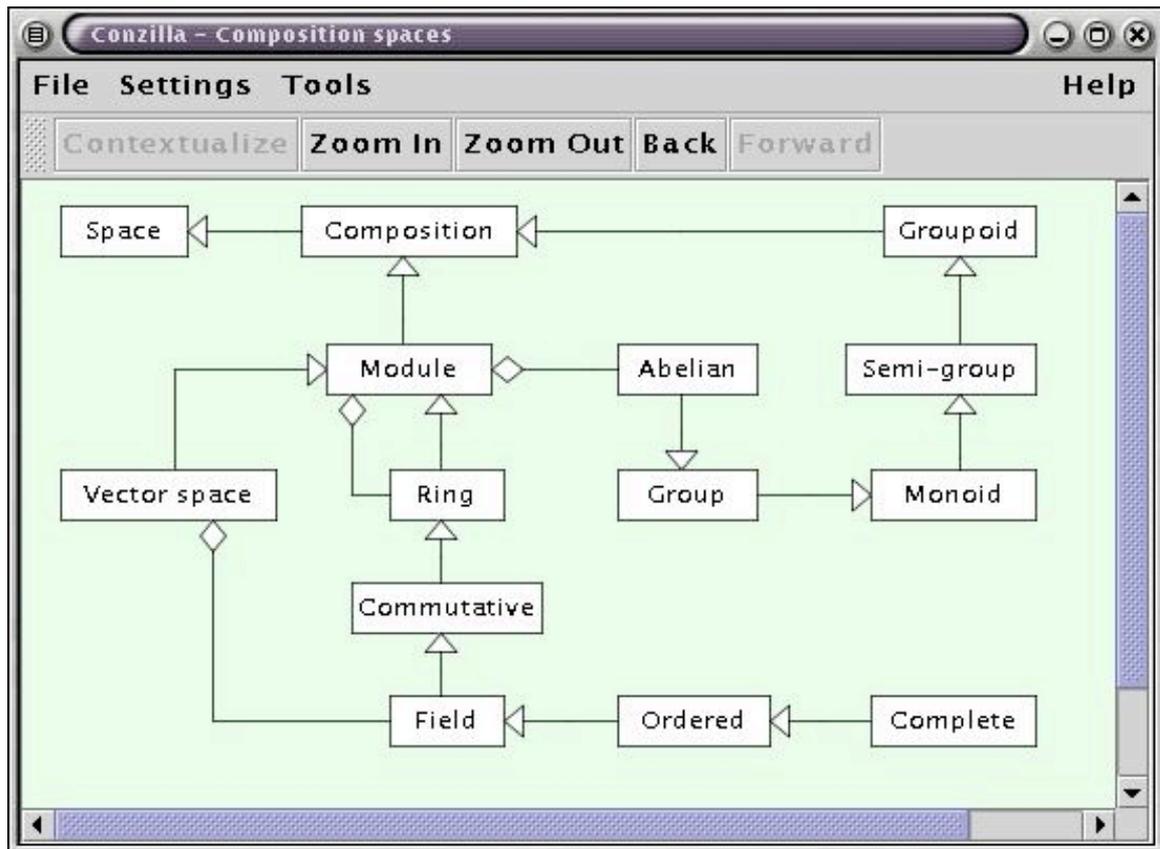


Fig. 7: A concept map showing different kinds of composition spaces.

Here we can see that Groupoids and Modules are special kinds of Composition Spaces, that Vector Spaces and Rings are special kinds of Modules, and that Semi-Group, Monoid, Group, and Abelian Group form a sequence of successive specializations of Groupoid. We also see from the Fig. 7 that a Module contains a Ring and a Vector space contains a Field, which in itself is a special kind of Commutative Ring. Fig. 7 also presents the concepts of Ordered Field and Complete Ordered Field

5.1.3. Displaying meta-data descriptions

Pointing to a concept box and hitting the arrow tangents, we can bring up various parts of the metadata list of descriptions of the corresponding concept. By pushing the \rightarrow key, we can move forward in the list, and by pushing the \leftarrow key we can move backward. Moreover, from any position in the list, by pushing the down-arrow, we can expand the list in the forward direction, and by pushing the up-arrow we can contract it again. And when we have found a piece of the description that we want to emphasize, we can leave it up on the screen and then move the cursor to another concept-box (or conceptual relationship arrow) and start “fishing for meta-data” in the new position. This gives us a lot of flexibility in controlling the simultaneous display of the various pieces of the different description lists for the entire collection of concepts and conceptual relationships that are displayed in the map.

A result of such a “meta-data fishing trip” is shown in Fig. 8. Here we see displayed some of the axioms for the concept of “Vector space” (nr 5-8), one

of the axioms for the concept of “Ring” (nr 1), and the definitions of Groupoid, Semi-group, Monoid, Commutative Ring and Field. We emphasize again that the meta-data of the different concepts can be manipulated independently of one another.

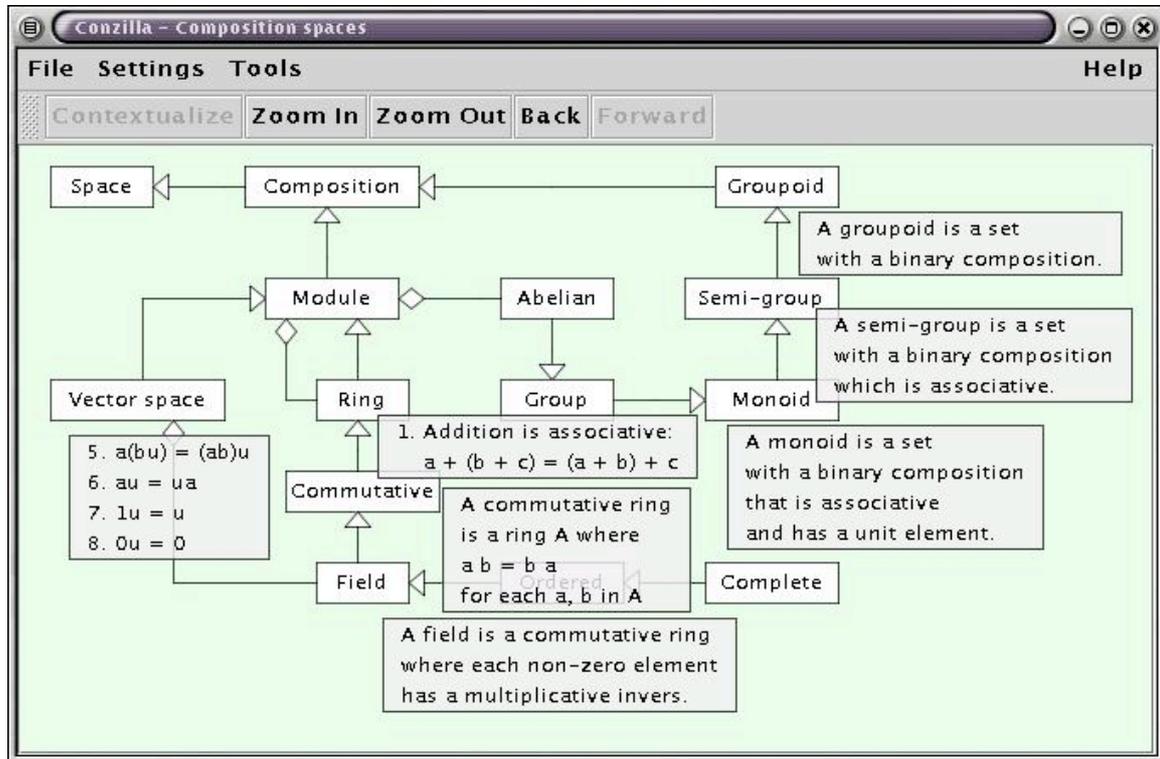


Fig. 8: Using the arrow-keys, we can bring up various pieces of metadata about the different concepts and conceptual relationships.

5.1.4. Viewing the list of content components through different filters

While changing the context is effected by choosing “surf” from the pop-up menu, viewing the content is achieved by choosing “view” from the same menu. This brings up the sub-menus that are displayed in Fig. 9. The three choices labeled “Aspects”, “Levels and “Resources” each correspond to different *filters* that display only the content components that are marked with the appropriate matching keywords . The choice of “Any”, which has been made in Fig. 9, causes the display of the entire list of components that match *any* of the three filters, while the choice of “Other” would display a list of the components that don’t match any of these filters.

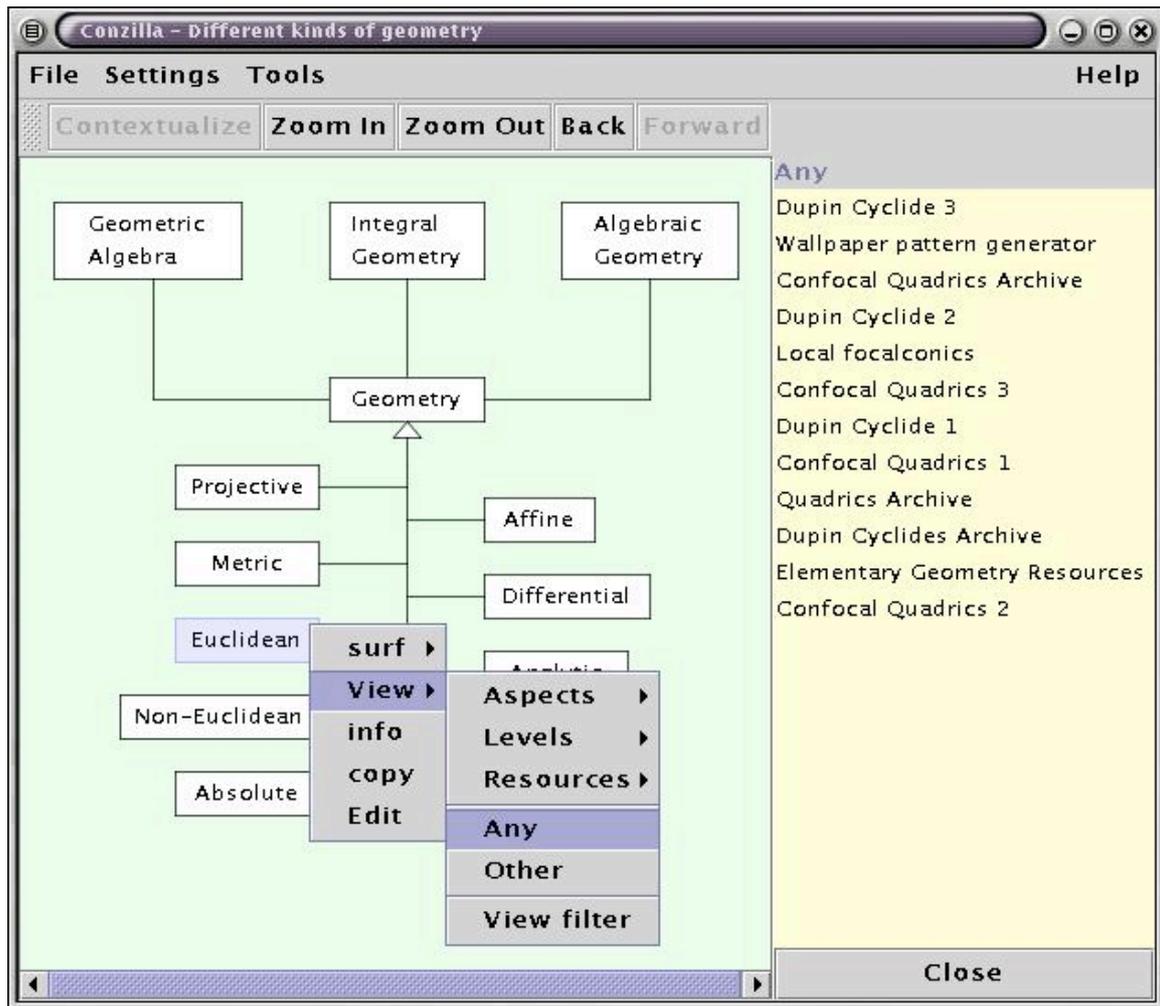


Fig. 9: Viewing the list of content components of the concept “Euclidean Geometry” that match any of the three filters “Aspects”, Levels” or “Resources”.

The display of a filtered list of content components is illustrated in Fig. 10. Here the content of “Projective Geometry” has been filtered through the Aspects-filter shown in Fig. 11. This filter map has been displayed by choosing “View Filter” from the third menu (= second sub-menu) shown in Fig. 10.

The Aspects-filter is constructed in such a way that each of the chosen aspects (What, How, Where, When, Who, Contact) is connected to each of the chosen levels (Elementary, Intermediate, Advanced, Expert). Hence these levels will appear as sub-filters (and sub-menus) under each one of the aspects. The greyed out parts of the filter menus of Fig. 10 correspond to combinations of keywords that are non-existent, i.e. not presently found among any of the components.

The Levels-filter is the “transpose” of the Aspects-filter, with each of the different aspects now appearing as sub-filters under each of the different levels. Taken together these two filters emulate the matrix-filtering technique discussed in [13].

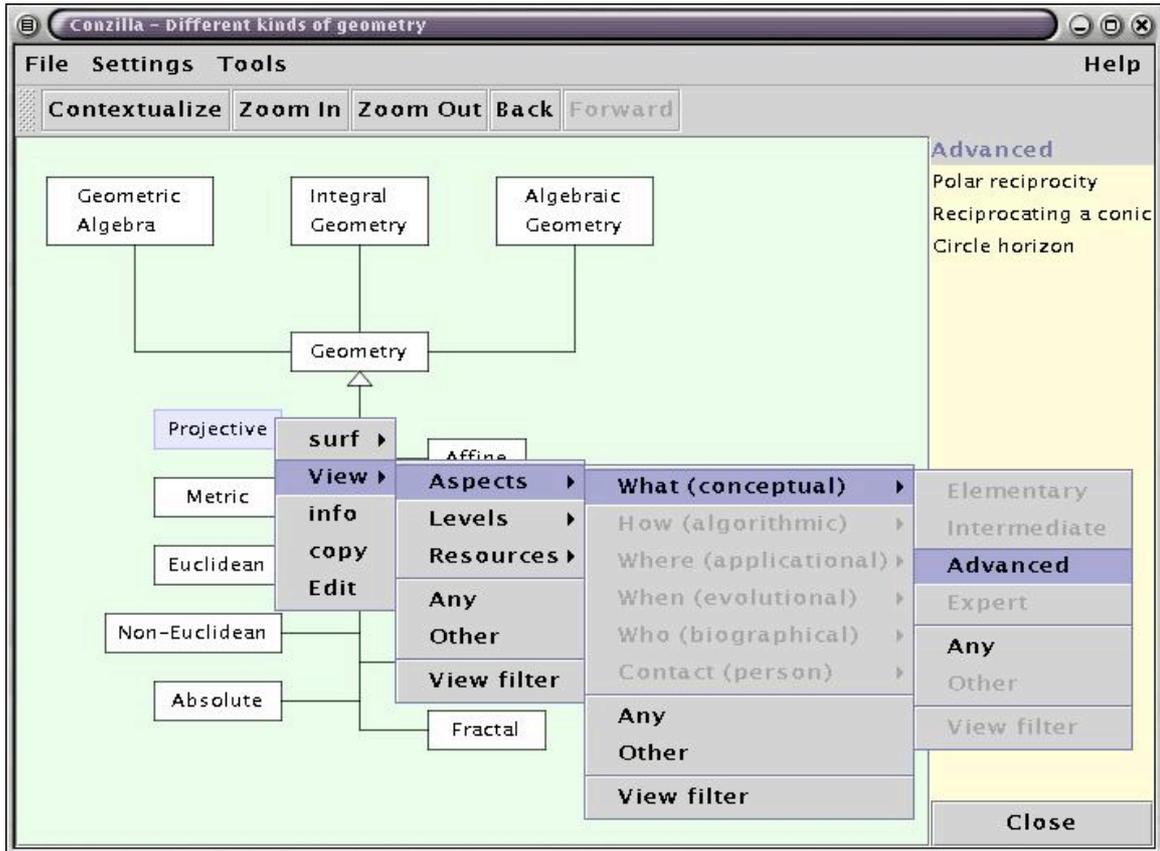


Fig. 10: Looking at the content of “Projective Geometry”, filtered through the aspect “What”, and the level “Advanced”.

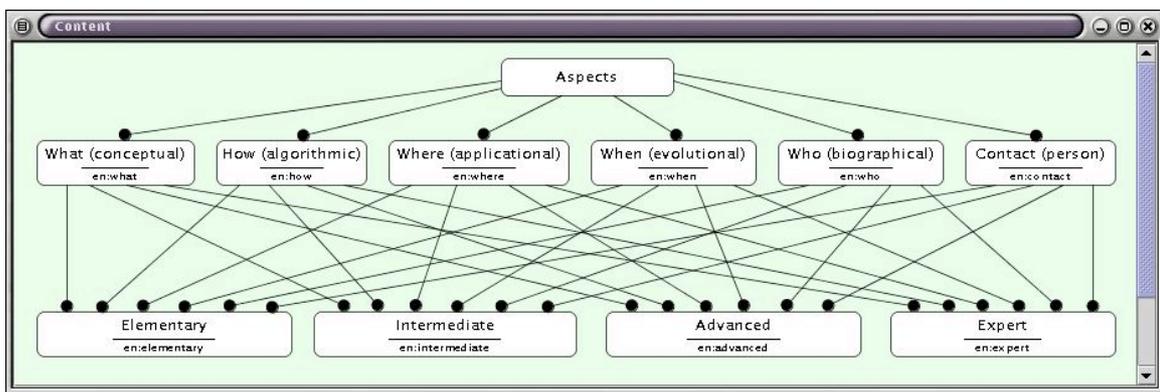


Fig. 11: A filter is controlled by a kind of Conzilla map (called a filter map).

5.1.5. Viewing the actual content of a component

Right-clicking on one of the listed content components and choosing “View” from the pop-up menu will display the actual content of this component in the web-browser of the user’s own choice. This is shown in Fig. 12. It is important to point out that the actual location of a component is well encapsulated from the rest of the Conzilla program. Hence a content component could be stored

locally (on the user's own computer) or anywhere on a computer connected to the Internet. Hence the Internet functions as a distributed component archive, with content that can be described (= tagged with meta-data) and associated with (= referenced by) any concept or conceptual relationship created by Conzilla.

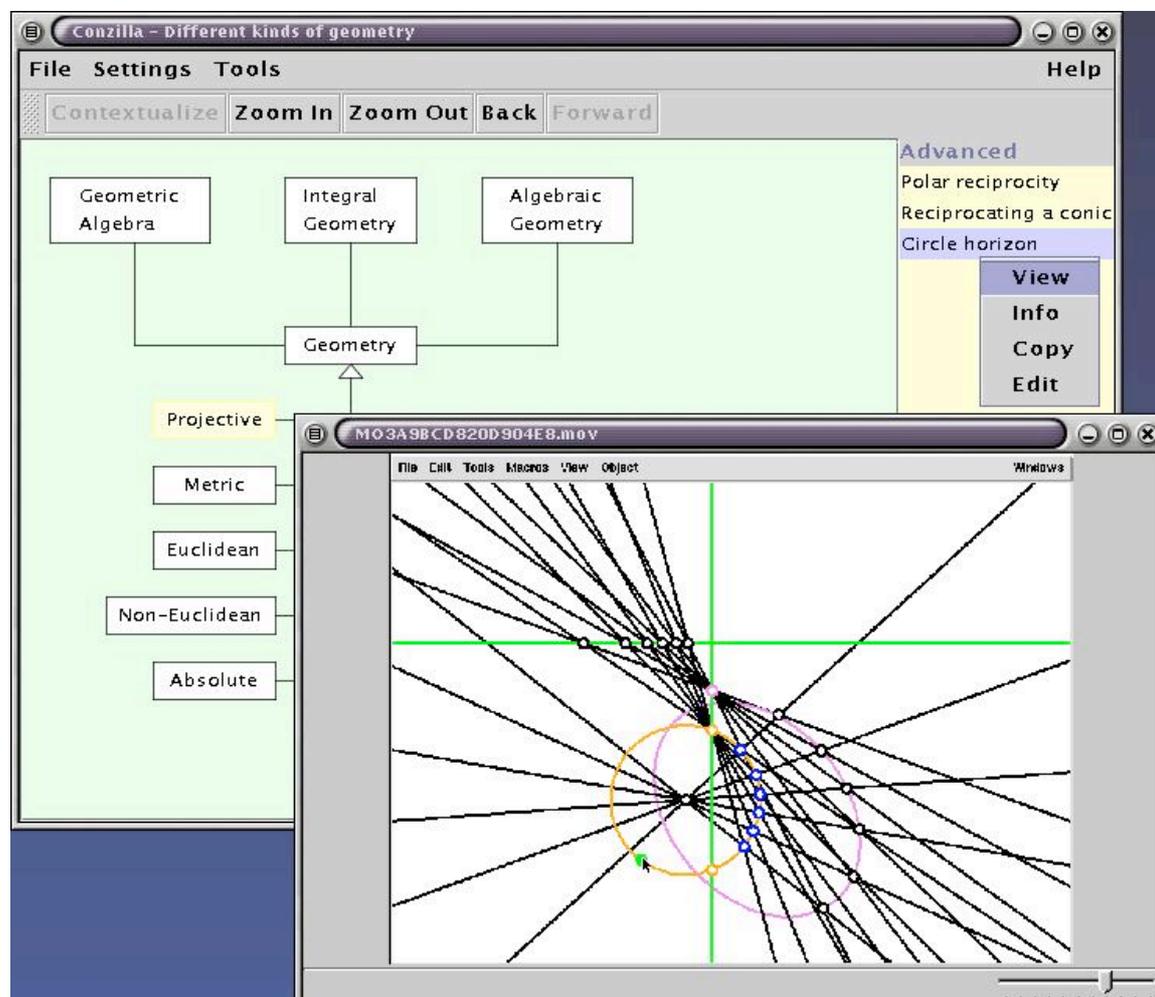


Fig. 12: Pointing to the component “Circle horizon” and choosing “View”, displays the content of this component, which in this case is a Quicktime movie. This movie is available at http://www.nada.kth.se/~amb/SnapzPro/Circle_horizon.mov.

5.1.6. Displaying the description of the content components

In Fig. 13 we have displayed the list of content associated with the concept of “Mathematics Learning Environments”, which is part of the “Mathematics Environment” context (= map). Note that since there are no filters associated with this collection of components, there are no sub-menus appearing under the corresponding “View” menu. Hence we are looking at the entire list of components associated with the concept “Mathematics Learning Environments”.

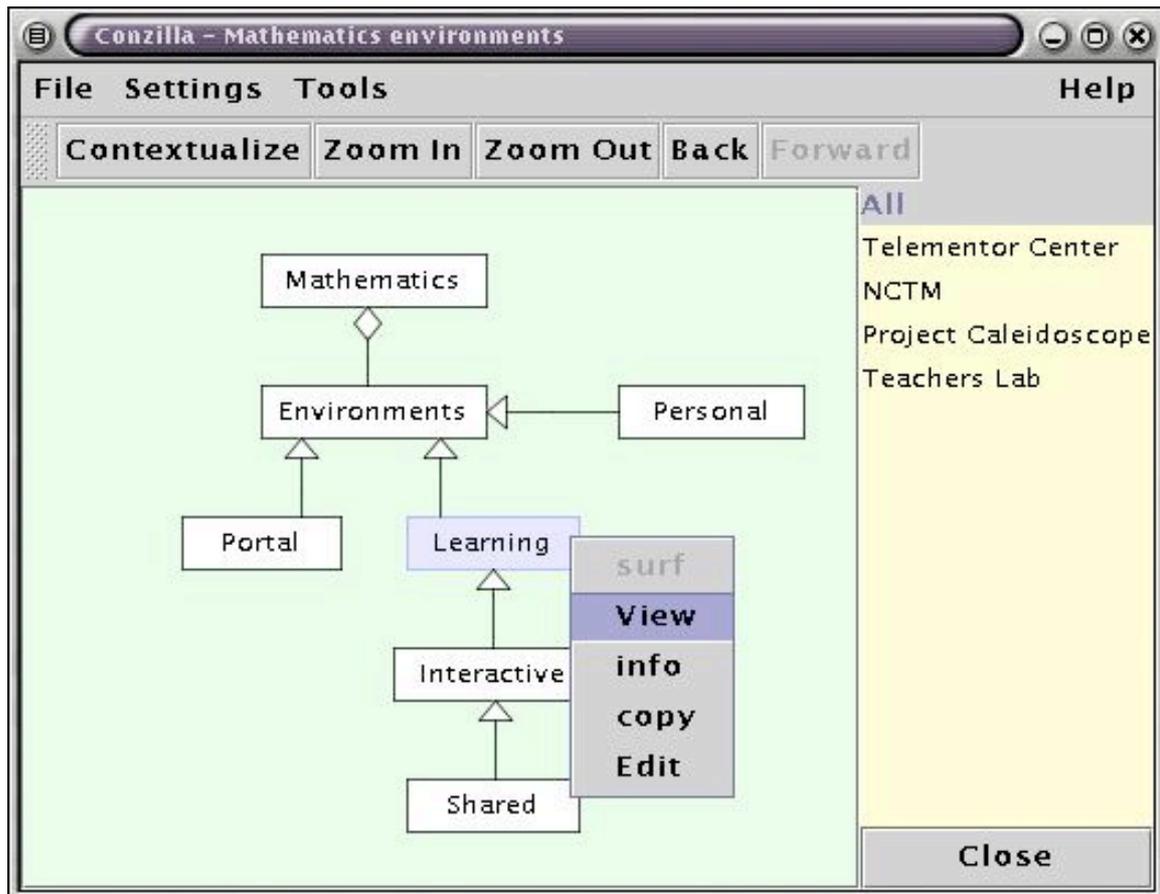


Fig 13: Viewing a list of all the content of “Mathematics Learning Environments”.

By pointing to an entry in this list and hitting space-bar (or the → key), the list of metadata descriptions for this entry can be manipulated just as before (Fig. 8).

In Fig. 14 we have displayed part of the descriptions for the components “NCTM” and “Teachers Lab”. In this figure we are pointing to the “Teachers Lab” entry in the content list. Therefore this entry becomes high-lighted together with its displayed metadata description. Pointing to the entry “NCTM” would high-light this name together with the description “National Council of Teachers of Mathematics”.

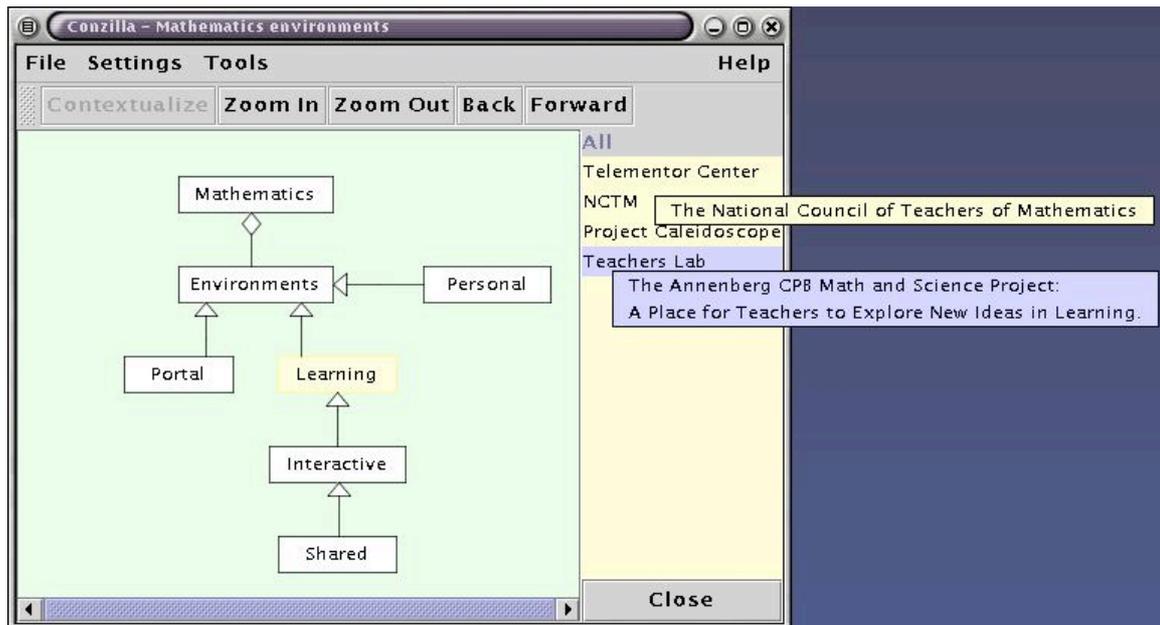


Fig 14: Looking at the descriptions associated with some of the components.

5.1.7. Design goals and educational applications of the VME

Mathematics contains at least 3 structurally different parts: concept formation, problem formulation and problem solution. Although the major part of a professional mathematics book is concerned with concept formation (= definitions), mathematics education (especially at the school level) is concerned mainly with solutions of pre-formulated problems. A basic idea that underlies the design of the VME is to make use of the computer supported multi-media based presentation technology in order to support the mathematical conceptualization process through visualizations of - and interactions with - mathematical concepts and formulas.

An important use of the VME will be as what could be termed a **“Skeleton in the closet website”** with the motto: “All you ever wanted to know about mathematics but were afraid to ask”. Here students can explore the “pre-requisite concepts” that they are supposed to know at e.g. the university level, and consequently do not dare to ask about during a lecture because of fear of loosing face in front of the teacher as well as their fellow students. In this way the VME can help to **overcome some of the transition difficulties** between the different levels of mathematics education, which is a growing problem that is gaining increased recognition both in educational and political circles.

The VME will be designed to be of use for students in general when preparing for an exam on a mathematics course. Moreover, it will be especially useful for the growing number of older students that have not (yet) managed to pass the exams of their mathematics courses. The overview and conceptual connectivity made possible by this way of presenting information should be a valuable aid in preparing for an exam – especially when the student is not presently following a live presentation of the relevant material.

An important idea that underlines the design of the VME is the conviction that no real questions can be answered in an automated fashion. In fact, it is only when the questions of the learner break the pre-programmed framework of the system that the real (= non-trivial) learning process begins. Hence the learner must be given access to live knowledge sources that can discuss the questions and guide the learning process. This is the idea behind the Contacts-node of the Aspects-filter of Figure 11. We are presently working on the design of a help-service system for Conzilla, where electronically certified knowledge sources will be available for the learner to contact in order to discuss issues that come up in connection with a certain concept⁶. In this system, teachers will be able to enter into learner-initiated online discussions about their favourite subjects with learners from all over the world that have navigated themselves into a place where some non-trivial questions have come up. In this way the VME will serve as a platform for “**non-trivial mathematical pursuit**” with a design that supports communication “man-to-man via machine”. This design will take advantage of the network in order to create a better match between teacher knowledge and learner interests than that which can be found within most educational systems of today. These ideas are discussed in more detail in [11] and [13].

The design of the VME has been started by Ambjörn Naeve as outlined above. However, the aim is that the future development should be carried out in a distributed manner, where a multitude of contributors (= knowledge gardeners) can link their own knowledge patches to patches created by others that they feel have something relevant to offer. For example a knowledge patch on the subject of Geometric Algebra⁷ is presently being brought into the VME by Mikael Nilsson, as part of his masters thesis project in mathematics under the joint supervision of Ambjörn Naeve and Lars Svensson⁸.

The **primary learner target group** of the VME are university students with an interest to improve the conceptual foundation for their mathematical studies by filling knowledge gaps that have been created during earlier parts of their education.

The **secondary learner target group** are high-school students with an interest to find out more about mathematical concepts and how they are related to each other.

The **primary teacher target group** is the educational reformists, struggling to make use of the unstructured plethora of educational raw material that is available on the Internet in order to create more interesting and stimulating learning experiences for their students.

⁶ This is the goal of a masters thesis project in interactive learning environments that is carried out at CID by Johan Olsson under the supervision of Ambjörn Naeve.

⁷ Geometric Algebra is a fascinating and little known area of mathematics, which has been championed by Prof. David Hestenes of Arizona State University. It holds the potential to revolutionize many parts of traditional mathematics education. See below (or http://modelingnts.la.asu.edu/GC_R&D.html) for further details on this subject.

⁸ Senior lecturer at the Mathematics department at KTH.

The **secondary teacher target group** is the teachers that want to continue their own education within the field, without necessarily taking time-out from the everyday duties of teaching.

The overall aim of the VME development project is to create a system that supports **first class mathematics education** for as many learners as possible. It should be in the common interest of universities and schools to collaborate in such an effort, since there is an increasingly felt need for customizable and individually adaptable learner support at all levels of the mathematics educational system.

5.2. LiveGraphics3D – making Mathematica graphics come alive

LiveGraphics3D is a Java applet written by Martin Kraus and licensed for commercial use by Wolfram Research which allows 3-D solids created by Mathematica[®] to be rotated interactively right in the middle of an HTML page.

To use LiveGraphics3D, you must use a web browser supporting Java 1.1 (e.g., Netscape Communicator[®] 4.0/4.5 or Internet Explorer[®] 4.0/4.5/5.0). You must also activate Java in the preferences or options menu of your web browser.

The LiveGraphics3D software is available for free downloading (for non-commercial use) at www.treasure-troves.com/buy/math/cdrom/live.html.

Using LiveGraphics3D, Ambjörn Naeve has created a large collection of surface models that illustrate concepts from classical algebraic- and differential geometry. These include the well-known quadric surfaces (ellipsoid, one- and two-sheeted hyperboloid, elliptic- and hyperbolic paraboloid), as well as the less familiar developable surfaces (tangential-, polar- and rectifying developable). They also include the Dupin cyclides, and various forms of so called Generalized Cylinders [9], which are a generalized form of lathed surfaces of special interest within the fields of Computer Vision and Computer Graphics.

There is also a Mathematica animation, which has been converted to interactive web graphics. It concerns the *double cylindrical point focus principle* [7], which is a way to focus planar wavefronts to a point by reflecting them in two parabolic cylinders (Fig. 2). It was discovered by Ambjörn Naeve and Lloyd Cross in 1976.

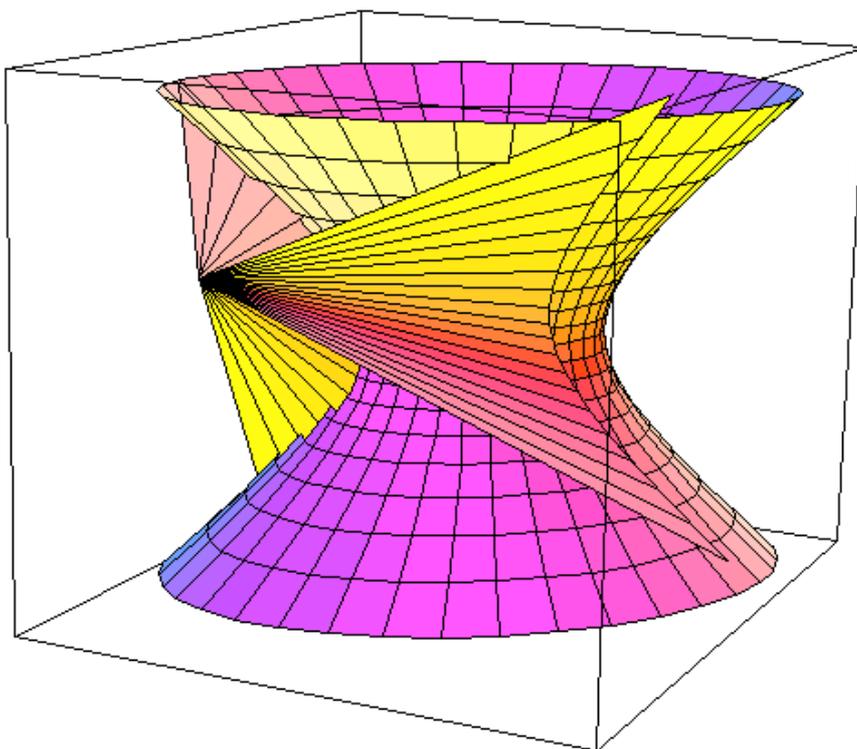


Fig. 15: The tangent cone from a point to a one-sheeted hyperboloid. The corresponding LiveGraphics3D java applet can be viewed at www.nada.kth.se/~osu/math/Geometry/Quadrics/one_sheeted_hyp_tang1.html.

5.3. Mathematical Component Archives

The mathematical ILE group at CID has started to experiment with different design principles for the creation of mathematical component archives that support distributed contributions and use. Olle Sundblad⁹ has created a tcl/tk-program called **dirli**ster, which creates opportunities for different ways of interacting with such an archive. Our test-archive is available at <http://www.nada.kth.se/cgi-bin/osu/dirliaster?math>.

This archive can be updated dynamically and the components are viewable under both Netscape and Explorer. Presently there is only graphics (and no text) on the html pages, but soon we will be putting in textual descriptions as well as formulas. We are presently investigating techniques to handle web presentation of mathematical formulas using MathML¹⁰. Traditionally this presents a well-known problem, which is usually solved by painful workarounds like e.g. screen dumps in the form of gif-images and the like.

⁹ A doctoral student at CID who specializes in virtual reality and network programming.

¹⁰ An emerging markup standard for mathematical formulas on the web. (See www.w3.org/Math.)

5.3.1. General goals of the MCA project

To apply general design principles for knowledge components to a concrete mathematics course should lead to results that are of value for mathematical didactics as well as for improving the design of efficient knowledge components in general. The directory-based MCA can be seen as the first step towards the following general goals:

1. To develop design principles and tools for the creation of archives of mathematical components that can be used by students in order to support their own assimilation of mathematical concepts and ideas, as well as by teachers as a resource for the composition and adaptation of learning modules to students with special needs, caused by e.g. lack of prerequisite knowledge or extreme cognitive profiles, as e.g. dyslexia or different kinds of visual or auditive impairments.¹¹
2. To create a generic structure for multi-media based knowledge components that can be filled with any kind of specific content, and which are designed in accordance with evolving international standards, such as IMS¹², RDF¹³, etc. Such a component-based content- and presentation structure - where the different components can cooperate through standardized interfaces - constitutes the basis of a new and rapidly emerging architecture for the organization and presentation of learning material which will form the backbone of the web-based education environments of the future.

5.4. CyberMath

CyberMath is a prototype of a shared 3D mathematical ILE developed by Ambjörn Naeve and Gustav Taxén¹⁴. The following two paragraphs are quoted from the CyberMath web site:

<http://www.nada.kth.se/~gustavt/cybermath/>.

It is well known that the current state of mathematics education is problematic in many countries. The Interactive Learning Environments group at CID is developing an avatar-based shared virtual environment called CyberMath, aimed at improving this situation through the presentation of mathematics in a new and exciting way. CyberMath is suitable for exploring and teaching mathematics in situations where both the teacher and the students are co-present and physically separated. The environment is built on top of DIVE, a toolkit for building interactive shared distributed virtual environments that support multiple simultaneous participants.

In the CyberMath environment, people (represented by avatars) can gather and share their experience of mathematical objects. When a person points to an object with the mouse, a red beam running from his avatar to the object appears in the 3D environment, similar to a laser pointer. Objects can easily be manipulated (rotated and translated) using the mouse. Since live audio is distributed as well, a person can point, act and talk - much as he/she would do in

¹¹ The ILE group at CID is presently engaged in a modeling project with HI (www.hi.se) and SIH (www.sih.se), where we are designing a Conzilla supported knowledge patch devoted to accessibility issues and cognitive profiling with a special focus on international standardization work within this field.

¹² See www.imsproject.org.

¹³ See www.w3.org/TR/REC-rdf-syntax.

¹⁴ A doctoral student at CID specializing in computer graphics and 3D systems programming.

real reality - as if the mathematical objects were hanging there in front of him/her. Hence, mathematics teachers are provided with a tool that integrates the best of both the virtual and the real world: virtual (mathematical) objects can be manipulated and discussed in a realistic way.

As mentioned earlier, CyberMath was presented (as a part of the Garden of Knowledge project) at the Siggraph 2000 conference in New Orleans . It was also presented at the CILT-2000 conference on new learning technology in Washington DC that was held in October 2000. Moreover, CyberMath has been accepted for independent presentation in the Educators program at Siggraph 2001 (Los Angeles, August 2001). As mentioned earlier, it has also been accepted for presentation at the ICDE-2001 world conference on open learning and distance education (Düsseldorf, April 2001).

On Oct. 5, 2000, we performed our first real distributed lecture in CyberMath, operating between Stockholm and Uppsala¹⁵. The teacher (Ambjörn Naeve) was physically located in Stockholm, and a group of 12 students were physically located in Uppsala. A tele-presence module was set up in parallel with the CyberMath system, which allowed teacher and students to have eye-contact during their interaction in cyberspace. This module was designed and operated by Claus Knudsen of the Media group at NADA as a part of the Communicative Spaces project within the Swedish Learning Lab.

CyberMath has recently been adapted to the VR-cube environment that is available at KTH (<http://www.pdc.kth.se/projects/vr-cube/>). There are still some minor problems to straighten out, but in principle it works great. We are planning a series of user studies to evaluate the differences of embedded VR versus desktop VR in this educational setting. CyberMath will also be used within a project called SHAPE (funded by the European Union) in which CID takes part (<http://cid.nada.kth.se/sv/forskning/eu-projekt.html>).

In the Wallenberg Young Scholars Program for the creation of the Virtual Museum of the Nobel e-Museum (www.nobel.se) there is a strong emphasis on the connection between young people's interest in a website and its ability to support advanced forms of animation and interaction. For the same reasons, we believe that CyberMath in a networked cave environment holds the potential to provide a high-tech front end which is interesting enough to create public interest and contribute to a more positive attitude towards mathematics - especially among young people. It could also provide a useful platform for developing various forms of interactive problem solving games - where the present violence-orientation of the gaming industry could be shifted towards an emphasis on cooperative problem solving skills, leading to future knowledge games that are designed to stimulate the learning interest of the gamers.

¹⁵ See pp. 47-50 below for an assessment of this experimental lecture.

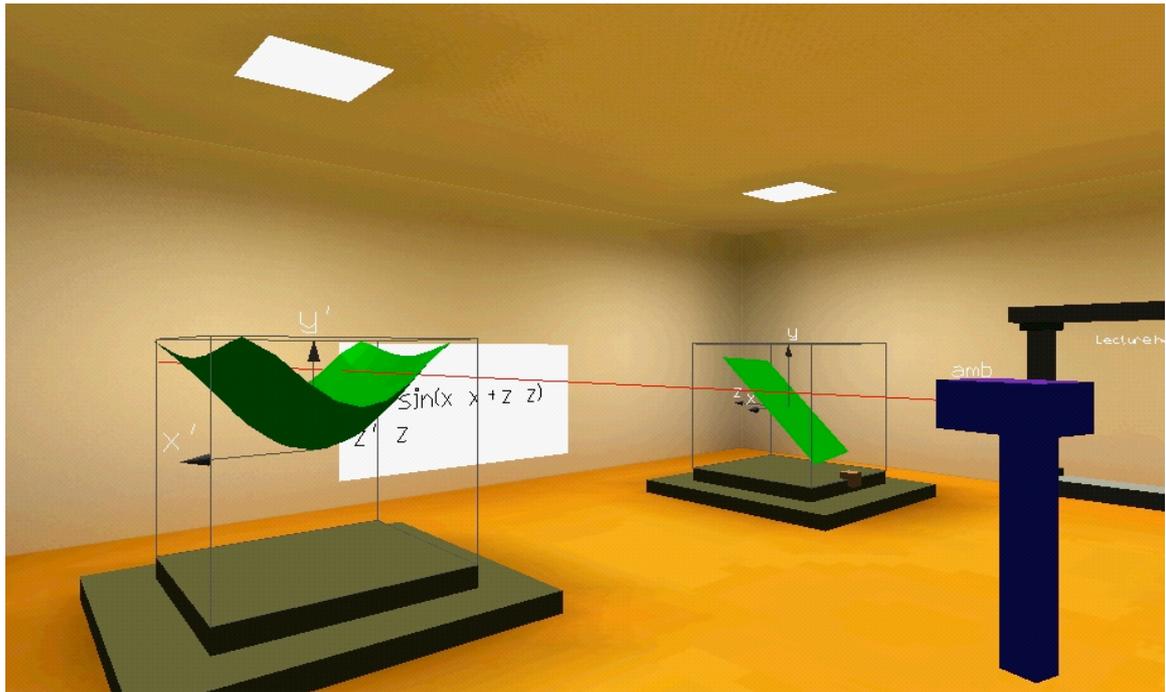


Fig. 16: The Transformation room: An avatar pointing to the image of the green plane under the mapping described on the wall.

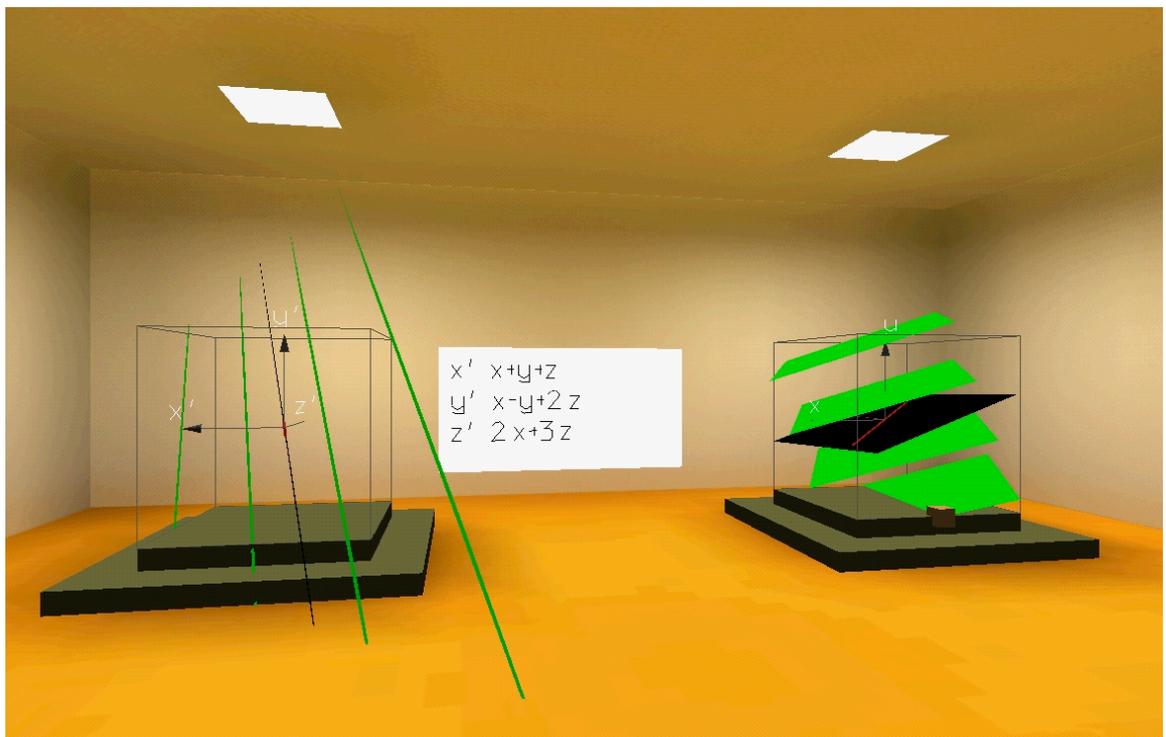


Fig. 17: The Transformation room: Determining the kernel of the linear mapping $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ on the wall. (Note: The equality signs are missing due to font problems)

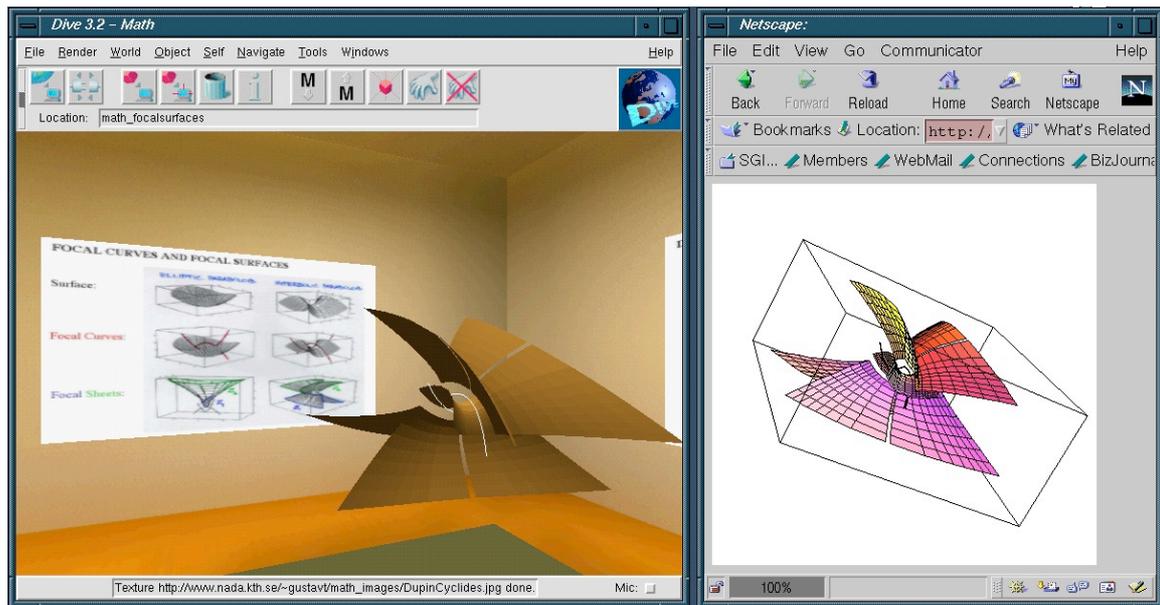


Fig. 18: Importing a Mathematica object (and connecting to the web archive)

Left-clicking on a mathematical object in the CybeMath environment, brings up a web browser and connects to a web archive where the corresponding object is stored as a component with graphics, formulas and explanations of various kinds. The object can then be manipulated (= rotated and scaled) directly in the browser using LiveGraphics3D.

It is a basic design decision to keep the CyberMath environment “formula-free” in order to avoid the neurotic reactions that mathematical formulas tend to trigger in many people at the first encounter. Since the geometric forms created by the formulas are experienced by most people as structurally interesting and aesthetically pleasing, these forms can serve as an incitement to overcome the “formula-neurosis” and penetrate the formidable language barrier that hides the magic and the beauty of mathematical formulas from public view.

6. The Swedish Learning Lab

The Swedish Learning Lab (SweLL) was set up during 1999 and officially created by a grant from the Knut and Alice Wallenberg foundation in November of that year. See <http://swedishlearninglab.org> for details of its organization.

6.1. Excerpt from the proposal summary¹⁶

The SweLL has been established as a platform for collaboration between Uppsala University, the Karolinska Institute and the Royal Institute of Technology in Sweden and Stanford University in the USA as well as with other partners within the Wallenberg Global Learning Network.

The SweLL has already started to act as one unity with a single governing body and one coordinating executive. The partners have built effective and interlinked teams for planning, assessment and technology, led by three directors. This proposal is in all parts the result of genuine cooperation between the three partners and their local organizations. The goal of the partnership is to make a significant contribution to the advancement of learning. The Swedish Learning Lab will engage in theoretical and empirical studies to explore new forms of collaborative learning supported by Information and Communication Technologies (ICT) in order to better understand how and when learning is improved and when it is not.

The proposal is aligned with the Stanford Learning Lab (SLL) and the WGLN mission. Through the collaboration and co-investments within this partnership we wish to establish common practices for research on learning, development of teaching and learning strategies and exchange of people, ideas, experience and expertise. Our approach is a scientific one. The experiments we intend to perform are guided by questions, which serve the purpose of giving a defined focus to each study, which can be measured and assessed.

The aims of this project proposal are to:

- explore the pedagogical possibilities of computer supported learning in specific areas of university education, e.g. through the creation of physical and virtual dynamic learning environments.
- develop new methodologies and strategies to promote the learning process.
- create new potentials for educators and students to mediate and access content through ICT in order to facilitate professional development.

¹⁶ For the full proposal text, see appendix.

6.2. New Meeting Places for Learning

A Pilot project within the Swedish Learning Laboratory (SweLL)

6.2.1. Summary

This pilot project will experiment with, and investigate, the potential of new information and communication technology (ICT) to put the student's learning in focus at universities and other learning organizations. Such a focus on learning requires the design of new learning activities and environments, which activate the learners, as well as systematic assessment of learning outcomes. This approach provides the basis for a critical evaluation of the effectiveness of the technology against the goal for which it is being used, namely, improved student learning.

6.2.2. Project overview

The three experiments proposed for the "New meeting places for learning" pilot project will perform work within the following areas:

1, *Distributed Interactive Learning Spaces (DILS)* – An experiment concerning the development of physical and virtual learning spaces supporting a wide spectrum of learning activities, local as well as distributed, supported by information technology. This experiment is a joint venture between KTH, UU, KI and Stanford.

2, Development and analysis of different *virtual collaborative techniques and tools* that can support the acquirement of not only theoretical knowledge, but also of "soft" knowledge, known to be important in professional life. Examples of this might be problem solving capacity and interpersonal communication skills.

These tools will be tested in the ISP-VL experiment at KI, UU and Stanford School of Medicine, with technical support of the infrastructure from KTH.

3, Experiments on *content archives, student portfolios* and *3D virtual learning environments*. These experiments will explore the implications of the use of digital resources created by students (personal portfolios) and teachers (content archives) as well as the impact of advanced 3D virtual environments as a medium to deliver course content to the students.

This will be performed in the Mathematics program of KTH, the Pharmacology course at KI and the Humanities area at UU but also as an exploratory experiment within the Linguistics course at UU and in cooperation with Stanford University. The 3D part of the experiment will be performed in collaboration between UU and KTH with important links to the ISP-VL experiment at KI and UU, mentioned above.

7. Archives, Portfolios and 3D Environments (APE)

The purpose of this experiment is to develop and test tools, principles and practices in the management of electronic portfolios, content archives and 3D communication and visualization environments for learning.

The APE experiment consists of 3 different tracks (A, B and C).

7.1. APE: Track A: Mathematics in Engineering Education

Principal Investigators:

Carl Gustav Jansson, DSV / KTH

Ambjörn Naeve CID / NADA / KTH

Leif Handberg MediaTechnology / NADA / KTH

Co-investigators:

Klas Karlgren, DSV / KTH

Olle Sundblad, DSV / KTH

Mikael Nilsson, CID / KTH

Matthias Palmér, CID / KTH

Richard Wessblad, DataDoktorn / CID / KTH

7.1.1. Goals

The goal of track A is to encourage and try out methods for the student use of e-folios to document and reflect over their learning process, including courses and the connections between them, beneficial for both further university studies and as a basis for life long learning.

Track A consists of two studies: one at the information technology program (study 1) and one at the media technology program (study 2) at KTH. The focus of the second study is more general while the first is more strongly focused on the structural and conceptual aspects of the mathematics education within the program. A common comparative study links these two studies.

7.1.2. Activity plan for study within the MSc program in IT at KTH

Study 1 will be structured as follows:

1. A mandatory task is introduced. Formally this task is attributed to the Introductory course for the whole program. The task is introduced in the beginning of the first year and is examined after the year is ended, however the work and teacher feedback should be done regularly over the whole year.
2. The task will have the character of a modeling exercise which reflects mathematical conceptual development during year one of the program. Following each of the two yearly KTH semesters, conceptual maps should be produced. The conceptual maps are assumed to be annotated in such a way that the students self reflection concerning understanding, perceived difficulty

and applicability is reflected. The more applied courses use and reference to mathematical concepts should also be captured.

3. On the introductory course some modeling knowledge is introduced together with certain notations and tools which is available for the students during the rest of the year.

4. All mathematics courses and applied courses during the first year will have lectures or exercises related to this task. All responsible teachers will be thoroughly familiar with the study. It will be discussed this spring during two planning workshops for the program.

5. The data collected during the year will consist of 2 times 150 conceptual maps. We will also make complementary observations during the whole first year.

Project plan

| | | |
|------|---------|---|
| 2000 | Jan-Jun | Detailed planning of study |
| | Jul-Dec | Activities within all courses on year one, mandatory exercises as basis for data collection |
| 2001 | Jan-Jun | -----"-----"-----"-----"-----"-----"-----"-----"----- |
| | Jul-Dec | Wave physics test, data analysis |

7.1.3. Excerpts from the Progress Report year 2000

Activities and progress (overview)

This section gives an overview of the main activities, outcomes and timing during the first six months. For further details, see section 4. At this stage of the project, activities are running according to work schedule.

Table 1: Activities overview in Study 1, Information Technology Program at KTH

| Activity | Outcome/Results | Realized/Timing |
|---|--|-----------------|
| Integration of the project activities within the framework of the IT program, year 1. | A positive and stable context for carrying out the study. | Spring |
| Seminars and conceptual modeling exercises for faculty involved in IT-program | Making the involved teachers more familiar with conceptual modelling Widening the range of the conceptual modeling exercises to include other subjects than mathematics on the IT-program. in a longer perspective. | Spring |
| Design of study-material for IT students | A paper on conceptual modeling of mathematical knowledge | Summer |
| Preparation of tasks | Short papers specifying the students tasks | Summer |
| Development of a lecture series | A specification to be integrated with the descriptions of the IT program curriculum | Summer |
| Design of evaluations for the study | Scheme of evaluations | Summer |

| | | |
|--|---|--------------|
| Software development | Usability tests by the end of the summer | Summer |
| | Start of two masters theses | Spring |
| Design of interactive web-support | Conceptual modeling activities and links to suitable interactive exercises (e.g. UML Tutorial). | Summer |
| Dissemination and outgoing activities such as: meetings, workshops, conferences, talks, presentations etc. (see section 6 for further details) | Contacts and future collaboration opportunities | January-June |

7.1.4. Current state of the project compared to the activity plan

Comments on activities in Study 1

Integration of project activities within the IT program framework, year 1.

Carl Gustaf Jansson has in dialog with the involved faculty and Ambjörn Naeve planned the study within the context of the IT program. The students will get a task split up into two parts (solutions to be submitted at the end of each term), a series of lectures integrated with the regular courses, study material on modeling in general and modeling of mathematical concepts, interactive study material and computer based tools for modeling. Formally the students will get 1 credit for this course out of the 5 credits for the course Introduction to IT.

Seminars and conceptual modeling exercises for faculty involved in IT-program. Ambjörn Naeve has on several occasions, illustrated the basic techniques of conceptual modeling in UML for the teachers involved in the IT-program at Kista, and shown examples of how this technique will be used in the mathematics courses. This has formed a good base for the current studies and created interest in widening the range of the conceptual modeling exercises to include all the different subjects on the IT-program.

Design of study material. Ambjörn Naeve has written a study material that presents conceptual modeling at a suitable level for this experiment. No material meeting the pedagogical requirements did exist. Apart from being a basis for this study, the produced material will have a much wider applicability.

Preparation of tasks. Carl Gustaf Jansson, Klas Karlgren and Ambjörn Naeve have designed and compiled a conceptual modeling task to be presented to IT students in the autumn and which will be the basis for their work within this study during 2000, 2001. The task should be solved in two steps (term 1 and term 2) The tasks are of three kinds, producing maps of high level conceptual maps, detailed maps on particular concepts and maps on the structure of proofs. The conceptual maps are assumed to be annotated in such a way that the students self reflection concerning understanding, perceived difficulty and applicability is reflected. A preparatory exercise has

also been produced, with the aim of training the students on modeling of the high school mathematics that is part of the curriculum during the introductory weeks.

Design of an evaluation scheme for the study. Klas Karlgren has in discussion with Carl Gustaf Jansson, Ambjörn Naeve and the SWELL assessment team developed an evaluation scheme including content and correlation analysis based on results from student tasks, observations and input from questionnaires.

Design of Interactive web-support. Ambjörn Naeve has together with Sofia Olsson and Kristina Edström at KTH Online developed a web-support for the conceptual modeling activities. This web support includes a UML tutorial (Interactive UML) produced by the Open Training company, some web-based animations produced within the Conzilla tool and some additional interactive exercises.

Software development. This involves improvements and extensions of the Conzilla tool. This work has proceeded according to plan. Mikael Nilsson has documented the code structure¹⁷. Ambjörn Naeve and Matthias Palmér has worked on aspect filtering. Richard Wessblad from DataDoktorn has worked with the the help-service system. Ambjörn Naeve have started up two master thesis projects for two KTH-students, Daniel Pettersson and Johan Olsson. These projects both represent extensions of Conzilla that will make the program more useful as an overall tool for knowledge management.

Student tasks split up into two parts (solutions to be submitted at the end of each term). The tasks are of three kinds, i) producing maps of high-level conceptual abstraction, ii) detailed maps on particular concepts and iii) maps on the structure of proofs. The conceptual maps are assumed to reflect students' understanding, perceived difficulty and applicability.

A comparative study of a course in wave-physics is planned (which links study 1 and 2). Three groups of students, from different engineering programs, with different experience in the use of organizational opportunities for reflection regarding the use of mathematics and conceptual modeling, will be tested after each version of this course. The task will be to select concepts from a mathematics concept map and judge the use in a wave physics context. Strategies have been developed for data-collection and analysis, where the focus will be to discern: What do the students reflect on during their studies and mathematical conceptual development?

7.1.5. Implementation of interactive content and appropriate tools

Mathematical Resource Components:

I. Constructing the components: Using programs like Mathematica, Projective Drawing Board and the Graphing Calculator, we have constructed

¹⁷ <http://conzilla.sourceforge.net/doc/conzilla-design/conzilla-design.html>

a number of mathematical resource components that illustrate mathematical concepts and relationships. Some of these components have been transformed into interactive web graphics and some have also been translated into CyberMath (the shared3D interactive learning environment for mathematics that has been created as a part of the APE-track-C project).

II. Archiving the components: There are different ways to archive mathematical components of different kinds - including the ones described above. A newly developed test-archive which can be updated dynamically and where the components are viewable under the common browsers is available at <http://www.nada.kth.se/cgi-bin/osu/dirliester?math> .

III. Interacting with the components: Exploring how to interact with the components, focusing on the ones constructed by using the Graphing Calculator, a program that is available today for the visual display of mathematical formulas. We have acquired 250 user licenses for the Graphing Calculator at KTH (to cover the teachers and students of the IT- and Media Technology program). The Graphing Calculator offers truly novel ways to interact with the components of a mathematics archive, where frozen animations can be downloaded and easily manipulated by users. This constitutes a very exiting graphical way of conducting mathematical discussions between the teachers and the students as well as between the students themselves. We have started to introduce this technique to some of the mathematics teachers at KTH, and we are planning to introduce it for the students of the IT-program in the spring of this year.

7.1.6. Educational evaluation/assessment results (Study 1)

During the fall of 2000 the 150 students of the IT program have been carrying out a conceptual modeling exercise as part of the "Intro to IT" course. The exercise is concerned with creating conceptual models based on the mathematics they have experienced in the IT-program curriculum. More specifically, three different concept maps are to be constructed: The first one should describe the overall relationships between the most important mathematical concepts that the students encounter in the mathematics courses, as well as the relationships between these concepts and their applications in other courses. The second concept map should focus on the function concept and describe the relations between the different types of functions that the students encounter in the mathematics courses, and the third concept map should describe the logical relationships between the different theorems that are presented in these courses.

The modeling activity was initiated by a lecture on conceptual modeling in UML given by Ambjörn Naeve on September 4. As a support structure for the modeling exercise, the students have had continuous access to the modeling web site and the interactive UML course described above.

The exercise stretches over the entire first year, and involves handing in two sets of these maps - one set by the end of the fall term, and a second set by the end of the spring term. The students have been allowed to work in groups of up to 4 persons.

At the moment the first set of concept maps have just been handed in. We are presently working out a way to present the results in a form which is as comprehensible as possible.

7.2. APE: Track C: 3D Visualization Environments for Learning

Principal Investigators:

Stefan Seipel DIS / UU

Yngve Sundblad CID / KTH

Ambjörn Naeve CID / KTH

Co-investigators:

Han Fei, DIS / UU

Gustav Taxén, CID / KTH

Olle Sundblad, CID / KTH

7.2.1. Goals of the CVEL– Project

- To increase the students ability to understand complex spatial and dynamic relationships in a variety of disciplines.
- To increase possibilities of collaborative interaction between students and shared exploration of course contents from remote teaching settings
- To improve the ability to access and retrieve course material by visually explore digital content archives

7.2.2. Activity plan year 2000 (Work packages)

WP 1: (DIS)

Selection of appropriate topics from the course in Interactive Graphical Programming (IGP) at Uppsala University. Specification of 3D content and interaction to be created and how to insert it into the avatar based learning environment. Specification of content types and means of interaction with this content in the Active Worlds ¹⁾ avatar based learning environment. In this working phase, teachers and a class of 40 students in the IGP course will be involved to help select appropriate content.

WP 2: (CID)

Selection of appropriate topics from the course in Mathematics at KTH Media Technology. Specification of 3D content and interaction to be created and how to insert it into the avatar based learning environment. Specification of content types and means of interaction with this content in the Active Worlds avatar based learning environment. In this working phase, teachers and a class of 30 students in the mathematical course will be involved to help select appropriate content.

WP 3: (DIS)

Implementation of interactive contents specified in WP1. Appropriate tools for implementation will be chosen depending on the type of content and type of interaction specified.

WP 4: (CID)

Implementation of interactive contents specified in WP2. Appropriate tools for implementation will be chosen depending on the type of content and type of interaction specified.

WP 5: (DIS/CID)

Specification and implementation of information representation landscapes. In this working phase, from selected areas of teaching (e.g. the above mentioned but also others) content is surveyed and catalogued, and visual representations of that content are created which can be explored by individuals in the Active Worlds environment. The work carried out comprises the design of a conceptual model for presenting those contents archives and the implementation of those in Active Worlds landscapes.

WP 6: (DIS/CID)

Identification of shortcomings of interaction mechanisms in Active Worlds. Specification of alternative means for interaction and navigation in Active Worlds (e.g. voice communication, gesture based navigation...). Specification and implementation of mechanisms to pick up content from an Active Worlds installation and to further process it using other applications.

WP 7: (DIS/CID)

In collaboration with the SwLL technology support team, implementation of appropriate functional extensions and mechanism defined in WP 6.

7.2.3. Excerpts from the Progress Report year 2000 - Results at DIS

Implementation of interactive 3D experiments:

In the second half-year period, program implementations have been carried out at DIS/UU with regard to the 3D interactive learning program for the *“virtual z-buffer experiment”* - based on the early prototypes generated in the first half-year period of the project. Functional specifications have been refined and implemented, and the experiment as such has been embedded into a virtual classroom environment framework. Under the course of this project, this virtual environment will serve as the main meeting place for student-teacher interaction also for other virtual experiments in the context of computer graphics lectures.

Data abstractions as required for networked collaborative performance of the *“virtual z-buffer experiment”* have been done, as well as implementation of network communication amongst many clients has been carried out. The virtual z-buffer tool can be used across multiple remote locations as of writing this report

Implementation study with a group of students:

Together with 32 students from the fall-course in “Interactive Graphical Systems”, a specification experiment was performed in order to identify needs and requirements to another 3D classroom experiment, which was already draft specified by students from the spring course. The interactive graphical performance analyzer is a learning medium to explore the relationship between graphical performance scene characteristics in graphical applications.

In this experiment, the previous draft specifications were turned into a formalized instruction to build such a interactive 3D performance analyzer tool. The formalized instruction was limited to a pure functional requirement specification (i.e. the students where told “*what*” to solve, rather than “*how*”). Therefore, in the student’s process of building such an application, their interpretation and understanding of the matter was expressed in a non-verbalized but still verifiable form. The students delivered eleven different implementations of the graphical analyzer program. The results were interesting, because some aspects which were not initially specified by the first student group as being essential to the problem, turned up in the delivered solutions. On the other hand, specific features identified as essential by the first group, were omitted in the implementations. This experiment gave some interesting ideas on how to build this interactive learning tool for the next generation of students.

7.2.4. Excerpts from the Progress Report year 2000 - Results at CID

Interactive 3D learning environment for mathematics:

In the second half of 2000, further program implementations have been carried out at CID/KTH with regard to the 3D shared virtual reality based mathematical learning environment called **CyberMath**, which has been described in an earlier progress report. An interface for controlling animations has been added to the exhibition halls, which includes starting and stopping as well as changing between displaying the animations in rendered or wireframe mode. Moreover, work has begun on the construction of a new exhibit called the **Transformation Hall**, where mathematical transformations from 3D to 3D can be studied in full generality. The implementation work on the transformation hall will go on during the spring of 2001 and educational testing will start during the fall of this year.

Educational evaluation/assessment results:

On 5th of October 2000, a preliminary assessment study was performed within the course “Interactive Graphical Systems”, given by Stefan Seipel during September and October 2000. Within this study, the CyberMath system was installed at a computer lab in Uppsala, and at KTH in Stockholm. It was supposed to be used by a number of randomly selected students from the above mentioned course.

Twelve students of the class where grouped into 6 groups with 2 persons in each group. Ambjörn Naeve held a virtual lecture on generalized cylinders. He was physically located in Stockholm, while in Uppsala three groups at a time were using the 3D computer graphics display to attend the lecture in the

Cybermath environment. Voice communication was facilitated through Cybermath as well as visual communication amongst the participants using their respective avatars. In addition to the pure avatar based communication, an “augmented reality interface” (presence production system) was provided by Claus Knudsen of the Media Technology group at NADA, in order to allow for simultaneous photo-realistic mediation between teacher and students.

The lecture, which took approximately 30 minutes, was performed twice for each three groups. The entire experimental lecture was video-recorded on both sides of the communication channels.

Prior to this virtualized lecture, the topic of generalized cylinders was also taught in form of a conventional lecture by Ambjörn Naeve to the other students of the course on 27th of September.

The goal of this first study was to deliver a proof-of-concept for the usage of the CyberMath system in the context of mathematics lecturing at the university level.

Assessment study of the distributed CyberMath lecture:

After the lecture, the 12 participants were given a questionnaire to fill in and return. The average age of the participants was 23.5 years. They all claimed that they work with computers on a daily basis.

The first part of the questionnaire consisted of 20 statements to be commented on a scale from 1 to 5. The meaning of the numbers were:

1 = I do not agree at all, ... , 5 = I agree completely.

Statements (followed by answer average):

1. It was easy to grasp the position and orientation of the avatars. (3.5)
2. I talked to other people in the cyberspace environment. (3.1)
3. I used my laser pointer frequently. (2.3)
4. It was easy to grasp what the other avatars were pointing to (with their laser pointers). (3.6)
5. It was easy to hear what the other avatars were saying. (2.6)
6. It was easy to grasp who was talking. (3.4)
7. The cyberspace environment felt inviting. (3.8)
8. I felt as if I was present in cyberspace. (3.5)
9. I felt as if the teacher and the other students were co-present in cyberspace. (3.6)
10. I talked to people in the real (physical) test environment. (4.2)
11. I explored the 3D cyberspace environment while the lecture was going on. (3.2)
12. I was disturbed by things that happened in the real (physical) test environment. (2.2)
13. It was easy to grasp the shape of the mathematical objects in cyberspace. (4.5)
14. The cyberspace environment made it easy to understand the material of the lecture. (3.9)

15. The lecture was interesting. (3.8)
16. The teacher presented the material well. (4.2)
17. I asked the teacher questions. (2.0)
18. I often ask questions during lectures. (2.0)
19. It felt more difficult to ask questions in the cyberspace environment than during a traditional lecture. (1.7)
20. It was easy to move my avatar in the cyberspace environment. (3.0)

There were also **two complementing questions of a commenting nature**:

- b. What was good with CyberMath?
- c. What was bad with CyberMath?

Comment: (in order to guide the interpretation of the answers): On two occasions during the lecture we experienced technical problems which forced us to reboot the system and restart the Cybermath environment from scratch.

These were the answers to the commenting questions:

1. It is new and exiting. The idea to move around in a room and give a lecture on its content is good.
2. It was difficult to move around, because there were often avatars and/or artifacts in the way. There were many things that distracted my attention, maybe because it was a new environment. It was hard to grasp what the teacher was pointing to.

1. It was easy to visualize the geometric curves. It was a relaxed form of teaching.
2. Some problems to move around and get a good viewpoint. The technique can obscure the content. Technical problems are never good, since a lot of time is wasted.

1. Good overview of the material, easy to explore the 3D shapes.
2. It was difficult to see what the other avatars were looking at. The sound delay (over the Internet) was a little too much to conduct a simple dialog.

1. Good visualizations that explain rather advanced subjects. A pleasant form of teaching.
2. More support for seeing where the other avatars were looking would have been nice. The environment felt a bit unstable - it was easy to move through walls etc.

1. Very good visualization of the different 3d-shapes! This made it much easier to understand how the different functions worked!
2. Maybe one shouldn't have to see the other avatars (except the teacher). It can be hard to see the mathematical objects if other avatars are standing in front of you.

1. Good idea! Nice with good visualizations!

2. It was bad that the other student avatars were visible. Also, the whiteboard was disturbing. It would be better if you could display it in a full-screen mode by pushing a button.
1. That you could get a good visual grasp of the mathematical shapes in 3D.
2. That it was too easy to move (very far) out of the room!
1. It was easier to understand the lecture since we could dynamically visualize 3D figures that are hard (or impossible) to visualize in a normal lecture.
2. Sometimes it was difficult to understand what the teacher meant, since he couldn't make use of gestures or other forms of body language.
1. Since the lecture was about 3D shapes, it was easy to understand their structure.
2. When you are new to this form of lecture, it feels a bit strange.
1. It was easy to understand the structure of the 3d-shapes. A good 3D environment to move around in.
2. It was hard to grasp what the teacher was pointing to.
1. You could see the shapes in a good way - as opposed to shapes being drawn on the blackboard.
2. Sometimes it was hard to handle the orientation. It was too easy to fly away far from the lecture room. It would be good if there were some more physical constraints in the simulation. Also, the environment was too sensitive to technical problems (our microphone didn't work).
1. The visualization of mathematical concepts made the lecture more interesting and easier to follow.
2. It was hard to discriminate between the different avatars.

Finally, the participants were asked a few questions aimed to give some information regarding their mathematical background as well as their prior experience in handling similar types of computer environments.

Question: How do you rate your own level of mathematical knowledge?

Answering scale: 1 = very low, 2 = low, 3 = rather good, 4 = good, 5 = very good.

Answering average: 3.8.

Question: How often do you work with 3D computer applications (e.g. Video games or visualization tools)?

Answering scale: 1 = never, 2 = a few times per year, 3 = a few times per month, 4 = each week, 5 = each day.

Answering average: 3.9.

As a **final question**, the participants were asked whether they believed that CyberMath would be a valuable tool to use in mathematics education at the university level. Out of the 12 participants 11 answered **yes** and 1 answered **no**.

Documentation:

We have created a VHS videotape of the recordings we made during this demo-lecture with CyberMath. Part of it is available in Quicktime format and is posted on CID's website at

http://cid.nada.kth.se/il/cybermathfilms/lecture_pstc_3.mov.

International Presentations and Publications:

CyberMath was presented in Washington DC on October 27 at the CILT-2000 learning conference (www.cilt.org/cilt2000) arranged by the Centre for Innovative Learning Technologies (CILT). A report from this conference in power-point format can be found at

www.learninglab.kth.se/library/presentations.

CyberMath was also invited for a special presentation at a workshop on modeling and visualization in Washington DC on October 25 arranged by EdGrid and NCSA (National Centre for Super-computing Applications) in connection with the CILT-2000 conference

(www.eot.org/edgrid/mvworkshop.html).

CyberMath has also been accepted for presentation (full conference paper + demo) at the 20:th ICDE world conference on open learning and distance education (www.icde.org) to be held in Düsseldorf on April 1-5, 2001.

Goal accomplishment

As for the first year of the project, we consider our goals as reached. In light of the various experimental pilot applications, which have been created throughout the project so far, we even consider us to have reached beyond our initial goal expectations.

The Learning Lab – Future Plans

The Wallenberg Global Learning Network has announced their intention to fund another set of projects intended to further develop the foundation that has been laid through the first initiative (partly) described above. The projects are planned to start during the second half of 2001 and have a duration of 2 years.

CID will participate in the PADLR project which is outlined below.

8. Personalized Access to Distributed Learning Resources

Coordinator: Prof. Wolfgang Nejdl, KBS, Hannover, Germany

8.1. Proposal Summary

Higher education faces problems of increasing student groups in combination with a shortage of resources for lecturing time. As time for student/teacher-interaction is reduced, students have to perform more self-studies than before. This calls for effective modularized and personalized learning materials, which help students to prepare for courses (pre-requisite courselets), support self-studying periods as well as own contributions during their courses, and provide additional background information as well as (project) documentation after their courses.

The driving vision for this project is a distributed "learning web infrastructure", which makes it possible to exchange/author/annotate/organize and personalize/navigate/use/reuse modular learning objects, supporting a variety of courses, disciplines and universities. Each of the PADLR subprojects therefore deals with a specific problem on the way towards our vision. Infrastructure, tools, courselets and archives will be designed/developed in accordance with international standards for modularization and metadata, and will be compatible across the PADLR project. We will specify how courselets are built (both from a technical and from an educational point of view), how they are organized and how they are exchanged and reused.

Several testbeds at our universities (a CS study program in Germany spread over 8 universities, the IT and the energy technology program at KTH Stockholm, courses in humanities, social sciences and bioinformatics at Uppsala University) will be the test and application area of our infrastructure, tools and courselets, providing a rich source of requirements, feedback and evaluation results for steering our project into the right direction.

8.2. Module: Infrastructure and Intelligent Services

8.2.1. Contributors

Stephan Decker (Stanford InfoLab/CS, Stanford),
Wolfgang Nejdl, Hadhami Dhraief (Hannover KBS),
Ambjörn Naeve, Mathias Palmér, Mikael Nilsson, Fredrik Paulsson
(CID/KTH Stockholm)

8.2.2. Working Title

Edutella: An Infrastructure for the Exchange of Educational Media

8.2.3. Problem Description, Research Aspects

Every single university usually already has a large pool of educational resources. These are under the control of the single entities (inter-university and outside), and it is unlikely that these entities will give up this control. Thus central-server approaches for the distribution of educational media are unlikely to happen. To facilitate exchange of educational media, we propose to develop a peer-to-peer exchange network for educational media. These peer-to-peer networks have already been quite successful for exchanging data in heterogeneous environments, and have been brought into focus with services like Napster and Gnutella, providing access to distributed resources like MP3 coded audio data.

However, pure Napster and Gnutella like approaches are not suitable for the exchange of educational media: Gnutella just uses the file name and path to index approaches. While this might work for files with titles like "Madonna - like a Virgin", it does certainly not work for "Introduction to Algebra - Lecture 23"

The educational domain, we thus see, is in need of a much richer meta-data markup of resources, a markup that is often highly domain and resource type specific. In order to facilitate interoperability and reusability of educational resources, we need to build a system supporting a wide range of such resources. This places high demands on the interchange protocols and meta-data schemas used in such a system, as well as on the overall technical structure.

8.2.4. Research Goals/Deliverables

To solve these problems we propose to use a meta-data based peer to peer system for educational resources. It is important to note that the peer-to-peer structures of systems like Napster and Gnutella are based on a client-to-client type of communication which is insufficient to guarantee access and stability of service. Although this kind of peer-to-peer will be an important option, the backbone of the system should be based on peer-to-peer between servers, which can be seen as a kind of distributed data repository.

Constructing such a system will pose two concrete problems: 1) defining the basic (abstract) model to use and 2) designing a sound implementation. When defining the basic system model, we will address the following areas:

1. Defining requirements for learning resource meta-data. We need to specify a basic model (such as RDF(S)), and an extensibility mechanism to be able to

incorporate varying forms of more advanced/specific meta-data (standard schemas like IEEE-LOM, learner models, OIL, area/subject specific meta-data, etc.). Focusing on RDF(S) will be strategically important in order to design our system to be compliant with the next generation (= semantic) web. Interoperability with existing web standards and easy integration into existing systems are crucial requirements to lower the entry level into the system. This is the fundamental part of defining the information handled by the Edutella system (KBS, CID).

2. Specifying requirements for the access to this meta-data. There is a fundamental need for addressing meta-data itself. In general, a consensus around semantic tags cannot be expected within any specific subject area - at least not in the short term. Hence we need to address the problem of allowing multiple sets of meta-data to be used for the same content. These sets must be separately identified. Another problem is the markup of meta-data itself, i.e. meta-meta-data. This is a fundamental part of using meta-data directly as an educational resource in itself, as is done in the Conzilla project (CID, KBS).

3. Defining the system's capabilities for importing/exporting other schemas/ontologies (Infolab)

4. Defining the meta-data handling capabilities (API, query language, updates). This is the fundamental part of accessing the Edutella system. (Infolab, CID).

When designing an implementation, we will need to make sure our approach is standards-compliant and portable. Our concrete goals will be to:

1. Develop a set of well-designed interfaces to the Edutella system that uses well-proven technologies such as XML over HTTP/DAV and CORBA. One important aspect is that setting up a basic Edutella server should be a very simple procedure. (CID, KBS)

2. Specify a number of supported addressing schemas. We need to address the problem of relocation of resources and servers. Especially in the educational domain, there is a fundamental need for persistent, location-independent resource identifiers. A two step procedure, where the resource identifier is separated from the actual retrieval process would allow easy relocation. Path URNs is such a solution, and IMS is working on other proposals in this direction. We need to define which such protocols that are to be supported in the system, and how the meta-data servers should cooperate in the resolving of such identifiers. (CID)

3. Implement an open-source base library with support for the basic meta-data model and the peer to peer networking details. (CID, KBS)

4. Design a modular/layered base system structure on top of this base library in order to support different types of extensions (e.g. an ontology inference engine) (KBS, CID, Infolab)

5. Implement a number of basic modules on top of the base library supporting meta-data querying, advanced meta-data caching, recursive searches etc. (Infolab).

6. Design/implement an open-source prototype meta-data server, supporting enough functionality to be immediately usable as a peer in an Edutella network (Infolab).

8.2.5. Interaction with other modules

Interaction with all modules building tools and functionalities, in order to define common standards, strong interaction with all modules working with test-beds in order to define requirements, and to use evaluations to drive development. Use research visits (2 weeks up to 3 months) (Hannover, Stanford, Stockholm) in order to integrate design and development within this module.

8.2.6. CID team

Ambjörn Naeve Ph.D., Senior researcher (Principal Investigator)
Mikael Nilsson, M.Sc., Ph.D-student (developer)
Matthias Palmér, M.Sc., Ph.D-student (developer)
Fredrik Paulsson, M.Sc., Ph.D-student (developer)

8.3. Module: Personalized access to educational media

8.3.1. Contributors

KBS, Hannover: Nicola Henze, Heidrun Allert , Jan Brase
DSV , KTH: Carl Gustaf Jansson, Klas Karlgren, Robert Ramberg
CID, KTH: Ambjörn Naeve, Mathias Palmér, Mikael Nilsson, Fredrik Paulsson
Stanford: Charles Kerns, Brad Osgood

8.3.2. Research Issues

This module attacks two major difficulties for teachers:

- the difficulty to share and reuse material produced by others for other contexts
- the difficulty to personalize and adapt existing material to a particular learning context.

We articulate seven main concerns:

- the commitment to shared standards, languages and tools to make sharing and reuse possible on a technical level (we here rely on results from the Module on infrastructure and content archives)
- the handling of large multimedia content archives, version handling and the handling of problems of structure and navigation (there are here clear relations to the modules on Infrastructures and Shared access to large archives)

- the sharing of simple models for course or courselet structures
- the modeling of domain knowledge, tasks and user competence as well as personal user preferences in such a way that it can form a basis for course structuring and personalization
- the personalization of content modules through annotations expressed as meta-data (there are also here clear relations to the modules on Infrastructure and Shared access to large archives)
- the personalization of courses or courselets through different ways to configure and modify structures of modules
- Adapting metaphors for course structure as well as kinds of visualizations for individual course modules

This project module will work with three major test-beds:

- mathematics and physics education within the Information Technology MSc program at KTH.
- computer science education within eight German universities including Hannover.
- mathematics education at Stanford.

Three studies are planned anchored in the three test-beds:

1. At KTH, Information Technology students will create conceptual models of the domain knowledge for introductory mathematics courses. Visualizations of mathematical concepts will be gradually integrated in all mathematics courses. Personalized mathematics courselets will develop in a bottom up fashion based on student designed conceptual models and additional context information. The annotations of the course modules + user models will be the core contextual information.

2. At Hannover, the KBS group will be content providers of the AI curriculum for a shared computer science curriculum based on large multimedia archives. The focus in Hannover will be on version handling for multimedia modules, navigation techniques based upon archives, conceptual models and meta-data + cognitive aspects of the adaptive functionality.

3. At Stanford, the emphasis will be on models for structuring courselets in mathematics looking at the easy reuse of subsets of courselets and personalization through editing and combinations of these subsets.

9. Geometric Algebra and Math Educational Reform

As mentioned above, the work of Naeve and his group within the field of mathematical ILEs is receiving increased international attention.

9.1. David Hestenes

Professor David Hestenes (Dept. of Physics, Arizona State University) is a well known physicist and a leading mathematics and physics educational reformist in the US. Hestenes is an internationally renowned authority on geometric algebra (or Clifford Algebra as it is also called), which is a coherent mathematical language which unifies a multitude of seemingly different mathematical systems. David Hestenes has spent the last 35 years advocating the use of geometric algebra as the basic mathematical tool in exploring physical reality. He is now spearheading the development of Geometric Algebra as a unifying language for mathematics and physics that can be used from the elementary level all the way up to the university education and research levels.

See [_](#) for further details on this ground-breaking project.

David Hestenes has also created a program for the reform of science teaching - based on conceptual modeling (See <http://modeling.la.asu.edu/modeling.html>). The Modeling Workshop program has been under development for more than a decade. Its approach to curriculum design and teaching methodology has been guided by a Modeling Theory of Physics Instruction, the focus of an educational research program directed by David Hestenes. The theory has been implemented in a practical Modeling Method for high school physics.

Based on the impressive results of this research, David Hestenes has been awarded several NSF (National Science Foundation) grants to further improve the modeling method and develop workshops to train other teachers to use it. Six week pilot workshops on three successive summers were found to exert a powerful influence on the teaching behavior of participants. This experience provides the foundation for the current nationwide program of Modeling Workshops funded by the NSF.

9.2. Modeling program excellence award

On Sept 25, 2000 Naeve received the following e-mail message from Hestenes:

////////// BEGINNING OF E-MAIL MESSAGE

Subject: [Fwd: U.S.Dept of Ed awards Modeling Instruction!]

Date: Thu, 21 Sep 2000 14:03:37 +0000

From: Jane Jackson jane.jackson@asu.edu

To: David Hestenes <Hestenes@asu.edu>

Sept. 15, 2000

The U.S. Department of Education announced on Sept. 12 that the Modeling Instruction Program at Arizona State University is one of seven K-12 educational technology programs designated as exemplary or promising, out of 134 programs submitted to the agency's Expert Panel last year. The list of the seven exemplary or promising programs can be found on the department's web site

<http://www.ed.gov/offices/OERI/ORAD/LTD/panel.html>.

I copied and pasted the following from the web site.

Cheers,
Jane Jackson

9.2.1. Exemplary & promising educational technology programs (2000)

The following seven programs were designated as exemplary or promising out of a total of 134 educational technology programs submitted to the U.S. Department of Education's Educational Technology Expert Panel. Selections were based on the following criteria:

- (1) Quality of Program,
- (2) Educational Significance,
- (3) Evidence of Effectiveness, and
- (4) Usefulness to Others.

Exemplary Programs:

- Challenge 2000 Multimedia Project
- Joint Venture: Silicon Valley Network (CA)
- Generation www.Y
- Olympia School District (WA)

Promising Programs:

- Maryland Virtual High School Core Models Project
- Montgomery Public Schools (MD)
- Middle-School Mathematics through Applications Program
- West Ed (CA)
- Modeling Instruction in High School Physics
- Arizona State University (AZ)
- One Sky, Many Voices
- The University of Michigan (MI)
- The WEB Project
- Montpelier School District (VT)

Promising:

Modeling Instruction in High School Physics uses computer models and modeling as a focal point to develop the content and pedagogical knowledge of physics teachers who then serve as local experts in the use of technology in teaching and learning science. In eight weeks of modeling workshops over

two summers, teachers revamp their current high school physics course to incorporate technology and insights of educational research. In the revamped course, instruction is organized into modeling cycles that engage students in all phases of model development, evaluation, and application. Students collaborate in planning and conducting experiments, use software to organize and analyze data, and present to the class a summary of their group's experimental interpretation, and findings.

Contact:

Jane Jackson

E-mail: jane.jackson@asu.edu

Phone: 480-965-8438,

Web site: <http://modeling.la.asu.edu/modeling.html>

////////// END OF E-MAIL MESSAGE

9.3. SIGGRAPH-2000: Geometric Algebra Course

Ambjörn Naeve has been collaborating with David Hestenes since 1997 on the use of Geometric Algebra as a tool to promote mathematics understanding in different ways. This collaboration also involves Prof. Lars Svensson at the Mathematics department at KTH, who recently (spring 2000) gave a course in Geometric Algebra which was very popular with the students.

In 1999 Naeve and Svensson were invited - through the mediation of Hestenes - to speak at the 5:th International Conference on Geometric Algebra and its applications to Physics and Engineering in Ixtapa, Mexico. Here Naeve presented his work on mathematical ILEs, which resulted in an invitation to join an international team of Geometric Algebra experts in presenting a 1-day course in GA at Siggraph 2000 in New Orleans, which was delivered on July 25 last year. The course was attended by approximately 450 students - including some of the top computer graphics programmers in the world - and it was a great success (see course comments below). In fact, the course presenters have an invitation to do the course again this year at Siggraph-2001. Here is a short description of the course:

9.3.1. Course Title: Geometric Algebra: New Foundations, New Insights

Organizers: Alyn Rockwood and Ambjörn Naeve

Length: Full Day, Advanced Course

Description: Geometric Algebra has been developed in the last decades from earlier pioneering mathematics of Grassmann and Clifford. It promises to stimulate new methods and insights in all areas of science and engineering, especially in computer graphics and related fields. This course first introduces and motivates the topic, and then provides example applications of interest for computer graphics professionals and researchers.

Speakers:

David Hestenes, Dept. of Physics, Arizona State University

Stephen Mann, Dept. of Computer Science, Waterloo University

Leo Dorst, Dept. of Computer Science, University of Amsterdam

Chris Doran, Astrophysics, Cambridge University.

Joan Lasenby, Signal Processing Group, Engineering Dept., Cambridge Univ.

Alyn Rockwood, Mitsubishi Computer Graphics Laboratory, Boston, MA.

Ambjorn Naeve, Dept. of Numerical Analysis and Computing Science, Royal Institute of Technology, Stockholm, Sweden.

Expanded Statement: Geometric Algebra is a new system for doing mathematics that unifies many different and redundant mathematical systems in current use. It is especially useful for handling geometric problems, since it allows for intrinsic, i.e., coordinate free, and dimensionally seamless descriptions of geometry. It has wide application in computer graphics, e.g., kinematics and dynamics, simplicial calculations (polygons, FEM), fluid flow, collision detection, hierarchical bounding spheres, boxes etc., quaternion splines on spheres, elastic deformations, curve and surface definition, vector fields and so forth. In all cases, new insights and improved algorithms invariably result. It can be seen as a new language for graphics, but it is much more, it is a new and fundamental language for the mathematics of computer graphics, modeling and interactive techniques.

After introducing the topic, using Java applets and interactive on-line and CD-ROM tutorials, we illustrate it with selected research and application topics.

Prerequisites: The main prerequisite is an active interest in mathematical fundamentals for computer graphics, and related areas. The course is listed as advanced because a reasonable level of mathematical maturity will ensure the maximal absorption of the breadth of topics presented. However, presentation will also be geared for those who want to glean the highlights, even without a full understanding of all the details. Typical concepts assumed will be vector calculus, complex analysis, quaternions, homogeneous coordinates, and linear algebra.

List of topics: Beyond the introduction of Geometric Algebra, we will present an improved model for generalized homogeneous space, fast intersection methods of planes and spheres, new ways to view conformal maps, projective geometry, methods for articulated systems and robotics, shape extraction and motion capture from scenes, elastic deformations and educational implications and approaches in general.

Some of the course comments:

- Fantastic! I just discovered an entire new field of math which is both fascinating and useful.
- The algebra software and particularly the illustrations really help the course!
- Very interesting course, apparently, the subject is too new for there to be many applications in graphics. However, there was a bit too much physics (I have a degree in physics!) which tended to obscure some of the potential

applications to Computer Graphics. That said, it was the best course so far, and I hope to see it again next year. Thanks!

- Excellent overall. New, deep ways of seeing rotations, translations and projective geometry = algebraically without singularities. Much of it already in quaternion, but this way is pedagogically superior.
- Somewhat abstract, but probably the most important course that I've taken in my 7 years at Siggraph!
- Excellent! The AV materials were particularly helpful in establishing the connection between the geometry and the algebra.
- WOW! Great way to present! Concepts via presentation demos.

9.4. MathXplor – a Virtual Mathematics Exploratorium group

Through his involvement with Siggraph, Naeve has established contact with computer scientist and geometric programmer Tamara Munzner (<http://www-graphics.stanford.edu/~munzner/>) who is part of the research group headed by Professors Pat Hanrahan and Terry Winograd at the Computer Science Dept at Stanford. Munzner is also a member of the (sadly shut down) Geometry Center development group (<http://www.geom.umn.edu>) headed by Bill Thurston. She is the custodian of an immense mathematical treasure - the collected video archive of the Geometry Center. She is also involved in an international collaboration committed to creating a virtual mathematics exploratorium on the Internet. This group (called MathXplor) includes a number of internationally well-known mathematicians (including Bill Thurston and John Sullivan). By demonstrating the mathematical ILE work that has been performed at CVAP and CID, Naeve has been invited to join the MathXplor group.

Within the Learning Lab APE project Naeve's group at CID is exploring how to handle mathematical resource components created by using a program called the Graphing Calculator which has been created by Ron Avitzur at PacificTech (<http://www.pacifict.com/>). He is based in Stanford and has close connections to Terry Winograd's research group. In Hestenes' opinion, the Graphing Calculator is the best program available today for the interactive visual display of mathematical formulas. It offers truly novel ways to interact with the components of a mathematics archive, where frozen animations (in the form of e.g. QuickTime movies) can be downloaded and brought back to life by dumping the corresponding files in the Graphing Calculator's formula window. The mathematical parameters can then be changed and explored and a new film can be recorded and uploaded to the archive. This constitutes a very exciting graphical way of conducting mathematical discussions between the teachers and the students as well as between the students themselves.

To convey an idea of what kind of visualizations that can be obtained with the Graphing Calculator, three examples are shown here. For some dynamic examples in the form of QuickTime movies, see <http://www.nada.kth.se/~amb/GraphingCalculator/confocal-quadrics.1.mov>

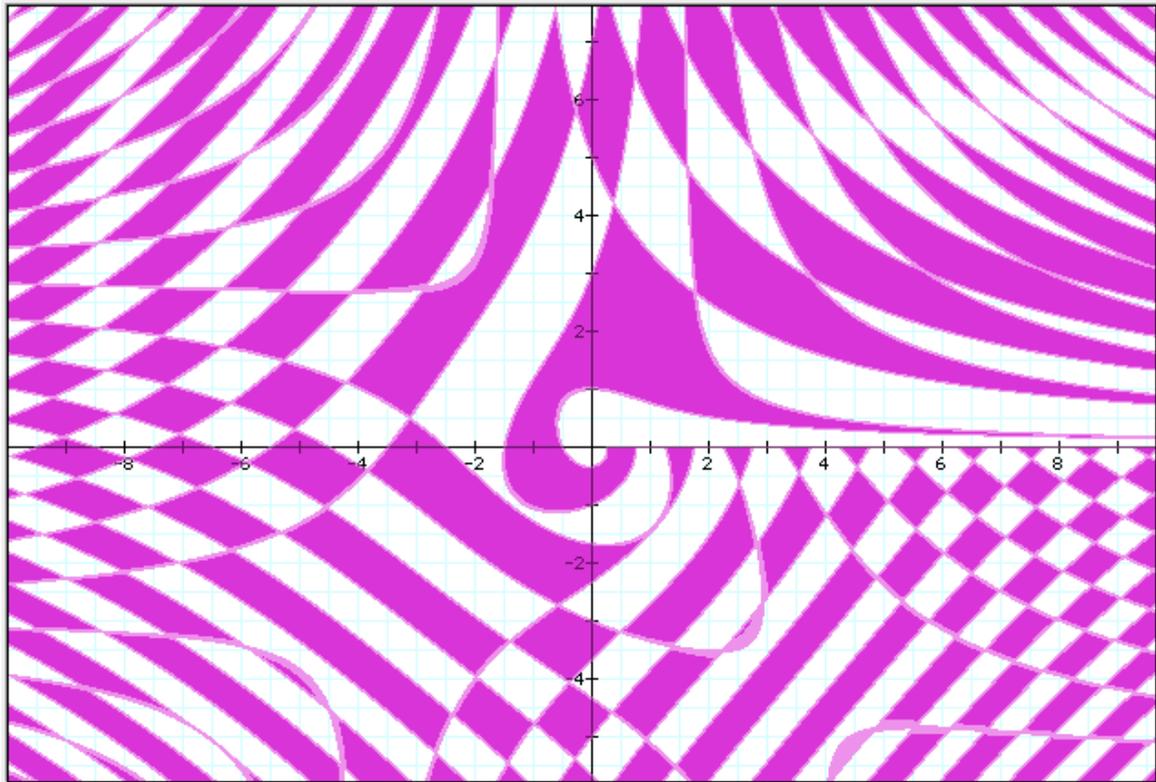


Fig. 19: Graphical plot of the points in the plane satisfying the inequality $\cos(r\theta) < \sin(xy)$.

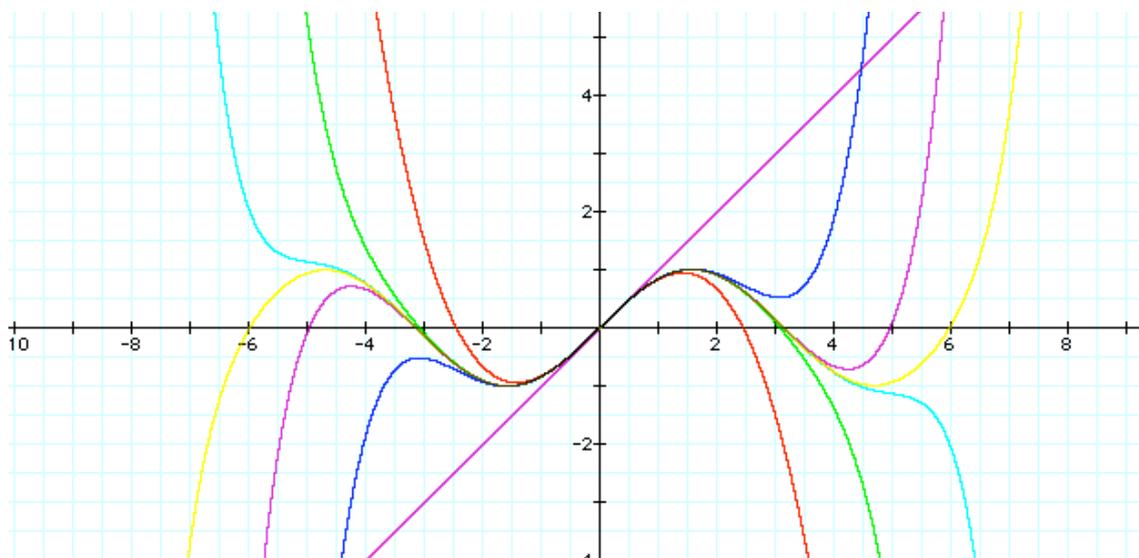


Fig. 20: The 7 first partial sums in the Maclaurin expansion of $y = \sin(x)$.
<http://www.nada.kth.se/~osu/math/Calculus/OneVariable/TaylorExpansion/sinx.taylor7.bin>

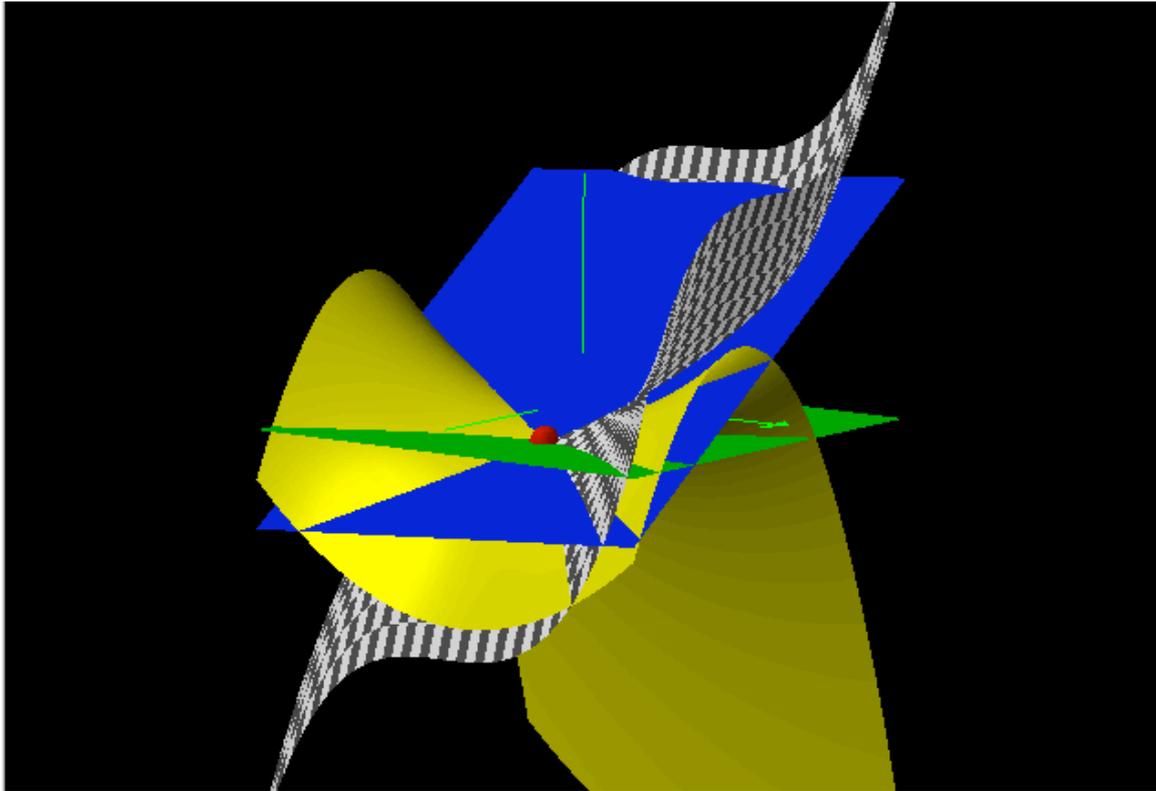


Fig. 21: The 3 first partial sums ($P_0 = \text{green}$, $P_1 = \text{blue}$, $P_2 = \text{yellow}$) in the Taylor expansion of $z = x^3 + y^3$ (checkered) around the red point. See www.nada.kth.se/~osu/math/Calculus/SeveralVariables/TaylorExpansion/x3plusy3.taylor12.mqv for a Quicktime movie of this expansion.

Within the APE project of SweLL we have acquired a site license for the Graphing Calculator and are planning to make use of it within the mathematics courses of the newly established IT-engineering program in Kista.

9.5. David Hestenes at KTH

Between Sept.13-19, 2000 David Hestenes visited KTH, and on Sept.13 he gave a talk at CID titled:

Computer Modeling Tool Kits to integrate & elevate K-20 mathematics & science

The talk presented Hestenes' modeling work and his plans for an international collaboration in order to build a series of software tools centered around a Geometric Algebra computational engine with a visual programming interface. This system will supply supporting tools for an educational reform that presents an integrated (K-20) approach to science and mathematics education from the beginning first grade all the way up through the doctorate level. The talk was documented by CITU in digital video format and will be available shortly as a streaming video from SweLL's website.

9.6. Educational Reform based on Software development

David Hestenes is presently involved in developing software tools that support this effort. <http://home.earthlink.net/~viridian1138/verdantium-hframe.html>
He is very impressed with the mathematical ILE work going on at CID, and he wants to get us involved in a strong international collaboration aimed at reforming the teaching of science and mathematics in a fundamental way. His collaborators include Jim Kaput and Jeremy Rochelle who are heading the SimCalc project.

9.6.1. The SimCalc Project

Here is a short description of the SimCalc project - taken from the project website <http://tango.mth.umassd.edu/>.

The Mathematics of Change is centrally important to living and working in a rapidly evolving democratic society. Problems involving rates, accumulation, approximations, and limits appear in everyday situations involving money, motion - virtually any situation where varying quantities appear. Yet only a small subset of students who take traditional calculus courses currently gain access to these concepts, and they do so only after a long series of prerequisites that eliminate 90% of their peers.

The SimCalc Project aims to democratize access to the Mathematics of Change for mainstream students by combining advanced simulation technology with innovative curriculum that begins in the early grades and includes powerful ideas extending beyond classical calculus. The principal directors of the SimCalc project are

James J. Kaput, Department of Mathematics, University of Massachusetts – Dartmouth <http://tango.mth.umassd.edu/website/team/kaput.html>

Jeremy Rochelle, Senior Cognitive Scientist, SRI International, Menlo Park, CA
<http://www.sri.com/policy/ctl/html/roschelle.htm>

Kaput and Rochelle are two of the major players within the field mathematical educational reform in the US. Hestenes has contact with both of them and is presently planning to collaborate with them on the K-20 educational reform project. He also wants to involve Naeve's group at KTH in this collaboration as shown by the following e-mail, which Naeve received on Sept. 21, 2000.

////////// BEGINNING OF E-MAIL MESSAGE

Subject: Possibility of international collaboration on Math Ed reform
Date: Thu, 21 Sep 2000 13:58:31 +0000
From: David Hestenes <Hestenes@asu.edu>
To: Jeremy Roschelle <Jeremy.Roschelle@sri.com>
CC: Ambjorn Naeve <amb@nada.kth.se>

Hello Jeremy,

Your message came at an opportune time. I have just returned from a week in Stockholm, where I visited a math education technology research group headed by mathematician/computer scientist Ambjorn Naeve at the Swedish Royal Institute of Technology. His group has very impressive software development resources, ideas and achievements.

Ambjorn is spearheading collaboration between his Institute and the Stanford Learning Lab, and he wants me to get involved. It might be natural to link up with SRI as well. I introduced him to SimCalc through your Web sites, so he is eager to meet you and Jim Kaput. He and I will be attending CILT 2000 in DC next month, so I hope you can arrange time for the three of us to meet (and, of course, include Jim if he is there).

Ambjorn is preparing a proposal for funding from the Wallenberg Foundation in Sweden. This is an outstanding opportunity to establish international collaboration on technology-based math education reform.
Cheers.....David

////////// END OF E-MAIL MESSAGE

9.6.2. Overall Strategy

By combining the work of our group (PDB, CyberMath, Conzilla) with that of Hestenes (GeoPad), Avitzur (Graphing Calculator) and Kaput & Rochelle (SimCalc) we have the possibility of achieving something with a really profound (and global) impact on mathematics education at all levels.

10. CILT and EdGrid

Two Powerful educational reform groups in the US

Through the mediation of Hestenes, Naeve was invited to present PDB, Conzilla and CyberMath at the CILT-2000 conference in Washington in October 2000

(http://www.cilt.org/html/cilt_2000.html). Hestenes also arranged an invitation for Naeve to participate in a workshop to Integrate Computer-based Modeling and Scientific Visualization into K-12 Teacher Education Programs, which was held in connection with CILT2000.

(<http://www.eot.org/edgrid/mvworkshop.html>).

10.1. The CILT 2000 conference and the M&V workshop

Invitation to attend a Workshop to Integrate Computer-based Modeling and Scientific Visualization into K-12 Teacher Education Programs

On Sept. 29, 2000 Ambjorn Naeve received the following email message:

////////// BEGINNING OF E-MAIL MESSAGE

Subject: Modeling and Visualization in Teacher Education meeting
Date: Fri, 29 Sep 2000 16:17:15 -0500
From: Lisa Bievenue <bievenue@ncsa.uiuc.edu>
To: Ambjorn Naeve <amb@nada.kth.se>

Dear Dr. Naeve,

I write to invite and encourage you to participate in the Workshop to Integrate Computer-based Modeling and Scientific Visualization into K-12 Teacher Education Programs on October 25-26, 2000, at the NCSA ACCESS center near the NSF in Ballston, VA. We hope that you will find the meeting location and dates convenient; they have been coordinated with the Fall CILT conference

(http://www.cilt.org/html/cilt_2000.html) in Washington, D.C., which I also strongly encourage you to attend. CILT features a Visualization and Modeling team and strand, during which we hope to continue the discussion and partnering efforts initiated at our meeting. We will be happy to coordinate travel arrangements so that you can attend both this workshop and CILT.

As a participant, your commitment is to

1. read and comment on several short papers in advance of the meeting, and
2. attend and actively participate in discussions throughout the 2-day meeting.

In exchange for this commitment we will cover your travel and hotel expenses.

The workshop is co-funded, and strongly encouraged, by The National Science Foundation and the US Department of Education. The mission of the workshop is to develop a research and implementation agenda for integrating modeling and visualization into teacher education. The format of the meeting will be structured to develop common understandings of

- The importance of modeling & visualization in doing science and mathematics, educational research on how modeling & visualization is and can be used in education (including cognitive, social, and equity issues),
- The issues of, and barriers to, implementation of these researched strategies in teacher education programs, and
- The role of national centers in the implementation of these strategies in teacher education programs.

The workshop is co-chaired by Marcia Linn (Berkeley & CILT), Lisa Bievenue (NCSA), Sharon Derry (National Institute for Science Education, University of Wisconsin College of Education), Mary Ellen Verona (Maryland Virtual High School), and Umesh Thakkar (NCSA). The coordinating consortium was the EOT-PACI (Education, Outreach, and Training - Partnerships for Advanced Computational Infrastructure, (see <http://www.eot.org>).

Key national leaders in teacher education, science education and technology have been invited to develop and present white papers on specific topics,

demonstrate how modeling and visualization tools are used in science, and lead sessions on specific issues for this focused workshop. Representative teacher educators, pre-service teachers, inservice teachers, school administrators, scientists, learning scientists, and educational technology researchers were invited to participate in the workshop. More information about the workshop goals, outcomes, schedule and topics is available at <http://www.eot.org/edgrid/mvworkshop.html>

We expect this workshop to have significant potential to direct funding priorities for educational research, development, and implementation in this area. We anticipate new cross-disciplinary collaborations and are planning for the next steps beyond the workshop in which partnerships will pursue additional funding to do the research and implementation called for in the workshop.

Welcome!

Lisa Bievenue NCSA Education Division, 605 E. Springfield Ave.,
Champaign, IL 61820PH: (217) 244-1993; FAX: (217) 244-1987;
EMAIL: bievenue@ncsa.uiuc.edu

////////// END OF E-MAIL MESSAGE

It turned out that Naeve was the only non-US researcher that had been invited to participate in the workshop. His presentations of the mathematical ILE-work at CVAP and CID (which focused on PDB, Conzilla and CyberMath) was highly appreciated and led to contacts with several people that expressed an interest in future collaboration.

11. References

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11.2. Supervisory Activities by Ambjörn Naeve

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- **[24]**: Olsson, J., *A Helpservice System for locating human competencies in conceptual browsing environments*, CID, work in progress.